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Algebraic Rate of Decay for the Excess Free Energy and Stability of Fronts for a Nonlocal Phase Kinetics Equation with a Conservation Law. I

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This is the first of two papers devoted to the study of a nonlocal evolution equation that describes the evolution of the local magnetization in a continuum limit of an Ising spin system with Kawasaki dynamics and Kac potentials. We consider subcritical temperatures, for which there are two local equilibria, and begin the proof of a local nonlinear stability result for the minimum free energy profiles for the magnetization at the interface between regions of these two different local equilibria; i.e., the fronts. We shall show in the second paper that an initial perturbation v_0 of a front that is sufficiently small in L^2 norm, and sufficiently localized that $\int x^2 v_0(x)^2 \, dx < \infty$, yields a solution that relaxes to another front, selected by a conservation law, in the L^1 norm at an algebraic rate that we explicitly estimate. There we also obtain rates for the relaxation in the L^2 norm and the rate of decrease of the excess free energy. Here we prove a number of estimates essential for this result. Moreover, the estimates proved here suffice to establish the main result in an important special case.

KEY WORDS: Phase kinetics; fronts; nonlinear stability.

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1. INTRODUCTION

The nonlocal and nonlinear evolution equation that we consider in this paper and in ref. 6 is

$$\frac{\partial}{\partial t} m(x, t) = \nabla \cdot (\nabla m(x, t) - \beta(1 - m(x, t)^2)(J * \nabla m)(x, t)) \tag{1.1}$$

where $\beta > 1$, $*$ denotes convolution and J is smooth, spherically symmetric probability density with compact support.

This equation first appeared in the literature in a paper[20] on the dynamics of Ising systems with a long-range interaction and so-called "Kawasaki" or "exchange" dynamics. In this physical context, $m(x, t)$ is the magnetization density at x at time t, viewed on the length scale of the interaction, and β is the inverse temperature. This introduction is not the place to fully explain the physical origins of the equation (1.1), and familiarity with them is not needed to understand our results or their proofs. Nonetheless, a few paragraphs on these origins are likely to provide a useful context.

Consider a lattice Λ which is, say, $\mathscr{Z}^n$ for some $n \geqslant 2$. At each site on this lattice we have a "spin" which is a random variable with values in $\{-1, +1\}$. There is given a probability density $\tilde{J}$ on $\mathbb{R}^n$, taken to be spherically symmetric, and the "length scale" of this density, say the square root of its variance, is much larger than the lattice spacing. A spin interacts with the "mean field" magnetization of all the spins around it, where the mean is computed using $\tilde{J}$, and the interaction is ferromagnetic, so that the energy is lower if the spin has the same sign as the mean field around it. Once an interaction energy and an inverse temperature β are specified, there is standard statistical mechanical procedure for specifying the equilibrium measures on the spin configuration space; these are the so-called Gibbs measures.

It is then natural to study Markov processes in the spin configuration space that are reversible for these Gibbs measures, and for which these Gibbs measures are invariant. One natural way to do this is to pick a spin at random, and then to randomly either "flip" it, or to leave it be, with probabilities depending on the energies of the flipped or unflipped configurations. Such processes were first investigated by Glauber.[17] Stochastic evolutions of this type are referred to as "Glauber dynamics".

Another, which leads to the equation considered here, is to pick a pair of neighboring spins at random and then to randomly either exchange their values, or to leave them be, with probabilities depending on the energies of the exchanged and unexchanged configurations. Stochastic evolutions of this type are referred to as "Kawasaki dynamics".[3] The chief difference between the two is that the total magnetization; i.e., the difference between the number of "plus" spins and "minus" spins, is conserved in the latter and not in the former.

Now, since we have taken the length scale of the interaction $\tilde{J}$ to be much larger than the lattice spacing, if one observes the system on the length scale of $\tilde{J}$, one sees it in a continuum limit. Instead of seeing individual spins, one sees a magnetization density $m(x)$, which is the local average of the spins near x. Since the spins have values in $\{-1, +1\}$, we have $-1 \leqslant m(x) \leqslant 1$. The evolution equation for $m(x, t)$ that one obtains in this limit is (1.1), with J related to $\tilde{J}$ by the scale change.

While this equation originated in ref. 20, it was not rigorously derived there, where instead, another process and a different equation were the main focus of study. Later, with x taking values in a torus T^n instead of $\mathbb{R}^n$, (1.1) was derived from the underlying stochastic dynamics in ref. 15. In any case, our investigation starts with (1.1), and it is independent of any of these derivations.

First, a fact that is basic to our work is that the Eq. (1.1) can be written in a gradient flow form. To do this, we introduce the free energy functional $\mathscr{F}(m)$ where

$$\mathscr{F}(m) = \int_{\mathbb{R}^n} [f(m(x)) - f(m_\beta)]\, dx + \frac{1}{4} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} J(x-y)[m(x)-m(y)]^2\, dx\, dy \tag{1.2}$$

where $f(m)$ is

$$f(m) = -\frac{1}{2} m^2 + \frac{1}{\beta}\left[\left(\frac{1+m}{2}\right) \ln\left(\frac{1+m}{2}\right) + \left(\frac{1-m}{2}\right) \ln\left(\frac{1-m}{2}\right)\right] \tag{1.3}$$

For $\beta > 1$, this potential function f is a symmetric double well potential on $[-1, 1]$. We denote the positive minimizer of f on $[-1, 1]$ by m_β. It is easy to see that m_β is the positive solution of the equation

$$m_\beta = \tanh(\beta m_\beta) \tag{1.4}$$

Then the equation can be written as

$$\frac{\partial}{\partial t} m = \nabla \cdot \left(\sigma(m)\, \nabla \left(\frac{\delta \mathscr{F}}{\delta m}\right)\right) \tag{1.5}$$

where the *mobility* $\sigma(m)$ is given by

$$\sigma(m)=\beta(1-m^2) \tag{1.6}$$

Then, formally one derives

$$\frac{d}{dt}\mathscr{F}(m(t))=-\int\left|\nabla\left(\frac{\delta\mathscr{F}}{\delta m}\right)\right|^2\sigma(m(t))\,dx \tag{1.7}$$

thus $\mathscr{F}$ is a Lyapunov function for (1.1).

This suggests that the free energy should want to tend locally to one of the two minimizing values, $\pm m_\beta$, and that the interface between a region of $+m_\beta$ magnetization and a region of $-m_\beta$ magnetization should have a "profile"—in the direction orthogonal to the interface—that makes the transition from one local equilibrium to the other in a way that minimizes the free energy.

There is considerable interest in the motion of these interfaces. Giacomin and Lebowitz[16] have provided formal arguments showing that on an appropriate length and time scale, in which one sees only the motion of "sharp" interfaces between regions of constant magnetization $\pm m_\beta$, the motion of these interfaces should solve a Helé–Shaw or Mullins–Sekerka free boundary problem. They also consider other scalings leading to interface motions such as Stefan problems. Formal work on the corresponding problem for the Cahn–Hilliard equation has been done in ref. 21, while rigorous results have been derived in refs. 1 and 8. The non conservative dynamics has been more completely investigated and many of the corresponding problems have been solved in refs. 11, 12, and 19. At the end of the introduction we shall go back to this.

To introduce the problem studied in this paper, consider a planar interface with m positive for $x_1 \geqslant 0$ and m negative for $x_1 \leqslant 0$, where x_1 is the first coordinate of x. Because the free energy is a Lyapunov function, the interface profiles that we expect to see in the system should, after an initial time interval at least, be such that they nearly minimize the free energy. To find the minimizer, one needs only consider functions m of the single variable x_1. Replace J by its marginal

$$J(x_1)=\int_{\mathbb{R}^{n-1}} J(x_1, x_2,..., x_n)\,dx_2\cdots dx_n \tag{1.8}$$

and simply use x to denote the single variable x_1. Then, it has been shown in ref. 14 that there is a unique function $\bar{m}_0(x)$ such that

$$\mathscr{F}(\bar{m}_0)=\inf\{\mathscr{F}(m)\mid \operatorname{sgn}(x)\,m(x)\geqslant 0,\ \lim_{x\to\pm\infty}\operatorname{sgn}(x)\,m(x)<0\} \tag{1.9}$$

Furthermore it is shown that $\bar{m}_0$ is an odd increasing function, and that

$$0 < m_\beta^2 - \bar{m}_0^2(x) \leqslant Ce^{-\gamma |x|}$$
$$0 < \bar{m}_0'(x) \leqslant Ce^{-\gamma |x|} \tag{1.10}$$
$$0 < |\bar{m}_0''(x)| \leqslant Ce^{-\gamma |x|}$$

for positive constants C and γ depending on J and β. The first two of these estimates are proved in ref. 14 and the third one in ref. 10.

The subscript 0 on the minimizer refers to the fact that the constraint imposed in (1.9) breaks the translational invariance of the free energy. For any a in $\mathbb{R}$, define

$$\bar{m}_a(x) = \bar{m}_0(x-a) \tag{1.11}$$

These functions $\bar{m}_a$ are the fronts whose stability is to be investigated here. Clearly $\mathscr{F}(\bar{m}_a) = \mathscr{F}(\bar{m}_0)$, so that $\bar{m}_0$ belongs to a one parameter family of minimizers of the free energy. There is another family, obtained by reflecting the previous one, because the free energy is also reflection invariant. However, these two families of minimizers are well separated in all of the metrics in which we shall work, and it suffices to consider only one of them.

Now consider an interface profile m that depends initially only on the single variable x. Clearly, it does so for all time and satisfies the evolution equation

$$\frac{\partial}{\partial t} m(x,t) = \frac{\partial}{\partial x}\left(\frac{\partial}{\partial x} m(x,t) - \beta(1-m(x,t)^2)\left(J * \frac{\partial}{\partial x} m\right)(x,t)\right) \tag{1.12}$$

This one dimensional equation for the evolution of fronts is the main focus of this paper. It has been shown in an unpublished paper by De Mottoni e Dal Passo,[9] that the evolution problem for (1.12) has a unique solution $m = m(x,t)$ with $\|m(\cdot,t)\|_\infty \leqslant 1$, locally Holder continuous in $\mathbb{R}\times\mathbb{R}_+$ provided the initial datum $m_0 \in H^1_{\rm loc}(\mathbb{R})$ and $\|m_0\|_\infty \leqslant 1$. More complete and detailed information will be presented here and in ref. 7.

The equation (1.12) not only has a Lyapunov function; it has a conservation law as well: For times t in any interval on which $m(t)$ is integrable,

$$\frac{d}{dt}\int (m(x,t) - \bar{m}_b(x))\,dx = 0 \tag{1.13}$$

for any b. Therefore, if one defines a in terms of integrable initial data m_0 for (1.12) by

$$\int (m(x,0) - \bar{m}_a(x))\, dx = 0 \tag{1.14}$$

one has for the solution

$$\int (m(x,t) - \bar{m}_a(x))\, dx = 0 \tag{1.15}$$

for all t or at least all t such that $m(s)$ is integrable for all $s \leqslant t$.

Now formally invoking the Lyapunov function and the conservation law, it is easy to guess the result of solving (1.12) for initial data m_0 that is a small perurbation of the front $\bar{m}_0$: The decrease of the excess free energy should force the solution $m(t)$ to tend to the family of fronts, and the conservation law should select $\bar{m}_a$ as the front it should be converging to, so the result should be that

$$\lim_{t\to\infty} m(x,t) = \bar{m}_a(x)$$

with a given in terms of the initial data m_0 through (1.14).

The main result obtained here and in ref. 6 is the following

Theorem 1.1. Consider initial data $m_0(x)$ for (1.12) such that

$$\int x^2 (m_0(x) - \bar{m}_0(x))^2\, dx \leqslant c_0$$

where c_0 is any positive constant. Then for any $\delta > 0$ there is a strictly positive constant $\varepsilon = \varepsilon(\delta, c_0, \beta, J)$ depending only on δ, c_0, β and J such that for all inital data m_0 with $-1 \leqslant m_0 \leqslant 1$, and with

$$\int (m_0(x) - \bar{m}_0(x))^2\, dx \leqslant \varepsilon$$

the excess free energy $\mathscr{F}(m(t)) - \mathscr{F}(m_0)$ of the corresponding solution $m(t)$ of (1.12) satisfies

$$\mathscr{F}(m(t)) - \mathscr{F}(\bar{m}) \leqslant c_2 (1 + c_1 t)^{-(9/13 - \delta)}$$

and

$$\| m(t) - \bar{m}_a \|_1 \leqslant c_2 (1 + c_1 t)^{-(5/52 - \delta)}$$

where c_1 and c_2 are finite constants depending only on δ, c_0, J and β and a is given by (1.14).

This paper is devoted to the proof of several key estimates used in the proof of Theorem 1.1. Further estimates, proved in ref. 6 are needed to complete the proof, which is done there. However, the estimates established here already suffice to cover an important special case, that we shall discuss at the end of the introduction.

Unfortunately, it does not seem possible to give a simple rigourous implementation of the heuristic argument given just before the theorem. There are several reasons for this, and the first of them concerns the norms involved.

The first thing that one might note about Theorem 1.1 is that the hypotheses concern L^2 norms of $m_0(x)-\bar{m}_0(x)$ and $x(m_0(x)-\bar{m}_0(x))$, while the conclusions concern the excess free energy $\mathscr{F}(m(t))-\mathscr{F}(\bar{m})$ and the L^1 norm of $m(x,t)-\bar{m}_a(x)$.

The following lemma, which is proved in the appendix in a slightly more general form, explains both why the L^2 norm is physically natural in the hypotheses of this theorem, and the relevance of the theorem to the problem of L^2 stability of the fronts. Throughout the rest of the paper, we write $\|f\|_2$ to denote the L^2 norm $(\int f^2(x)\,dx)^{1/2}$ of a function f, and $\langle f, g\rangle_{L^2}$ to denote the corresponding inner product.

Lemma 1.2. For any constant κ, there is are constants $\delta=\delta(\kappa)>0$ and $C=C(\kappa)<\infty$ such that

$$\frac{1}{C}\|m-\bar{m}_a\|_2^2 \leqslant \mathscr{F}(m)-\mathscr{F}(\bar{m}_a)\leqslant C\,\|m-\bar{m}_a\|_2^2$$

whenever $\|(m-\bar{m}_a)'\|_2\leqslant\kappa$ and $\|m-\bar{m}_a\|\leqslant\delta$, and $\bar{m}_a$ is any front that minimizes $\|m-\bar{m}_a\|_2$.

Note that whenever $\|m-\bar{m}_b\|_2<\infty$ for one value of b, it is finite for all b. Moreover, it clearly holds that

$$\lim_{b\to\pm\infty}\|m-\bar{m}_b\|_2=\infty$$

and thus, since $\|m-\bar{m}_b\|_2$ depends continuously on b, there is a value of a such that

$$\|m-\bar{m}_a\|_2=\inf_{b\in\mathbb{R}}\{\|m-\bar{m}_b\|_2\}$$

Moreover, we shall show in Section 2 that when $\|\bar{m}_a - m\|_2$ is sufficiently small, this minimum is attained *uniquely* at a. Since we shall be working throughout the paper in such a neighborhood, at each time t there is a priveleged front $\bar{m}_{a(t)}$ such that

$$\|m(t) - \bar{m}_{a(t)}\|_2 = \inf_{b \in \mathbb{R}} \{\|m(t) - \bar{m}_b\|_2\} \tag{1.16}$$

and $a(t)$ is a well-defined function of t since the minimum is uniquely attained.

Many of our estimates in these papers concern $m(t) - \bar{m}_{a(t)}$, and we now make the following convention: Whenever some solution $m(x, t)$ of (1.12) is under discussion:

$$v(x, t) = m(x, t) - \bar{m}_{a(t)}(x) \tag{1.17}$$

where $a(t)$ is given in (1.16), and moreover

$$\bar{m}(x) \qquad \text{denotes} \quad \bar{m}_{a(t)}(x) \tag{1.18}$$

Whenever v or $\bar{m}$ appear in what follows, this convention is being used.

The equation (1.12) has smoothing properties, explained in Section 2, that make Lemma 1.2 applicable to the solutions considered in Theorem 1.1. Hence it follows from the decay of the excess free energy established in Theorem 1.1 that

$$\|m(t) - \bar{m}_{a(t)}\|_2 \leqslant c_2(1 + c_1 t)^{-(5/52 - \delta)}$$

While the L^2 norm is physically relevant because it essentially measures the excess free energy, the L^1 norm is physically relevant because of the conservation law. In fact, in order to use the conservation law to show that

$$\lim_{t \to \infty} a(t) = a \tag{1.19}$$

we need to show that

$$\lim_{t \to \infty} \|m(\cdot, t) - \bar{m}_{a(t)}\|_1 = 0$$

The point is that if one can show that the excess free energy converges to zero, all one can conclude is that

$$\lim_{t \to \infty} \|m(\cdot, t) - \bar{m}_{a(t)}\|_2 = 0$$

The free energy functional is translation invariant, and so it gives no information on $a(t)$. In fact, it is not at all *a priori* clear that $a(t)$ even stays bounded. To control $a(t)$, we need L^1 bounds. Note that since we have the *a priori* bound $\|m(\cdot, t) - \bar{m}_{a(t)}\|_\infty \leqslant 2$, we have

$$\|m(\cdot, t) - \bar{m}_{a(t)}\|_2^2 \leqslant 2\|m(\cdot, t) - \bar{m}_{a(t)}\|_1$$

so that L^1 norms control L^2 norms in this problem, but not *vice-versa.*

In fact the proofs of the L^1 and L^2 parts of Theorem 1.1 presented in these papers are highly interdependent: We shall obtain good L^2 control only for times t such $\sup_{s \leqslant t}\{|a(s)|\}$ is not too large. To show that this is the case for all t, we need good L^1 control. On the other hand we use L^2 control as one ingredient in obtaining L^1 control. Moreover, the L^1 control we need is somewhat difficult to obtain in this problem due to a lack of monotonicity as we explain below.

The first paper[2] on this problem was by Asselah, who proved that for small L^2 initial data

$$\lim_{t \to \infty} \|m(t) - \bar{m}_{a(t)}\|_\infty = 0$$

The proof was based on a compactness argument exploiting (1.7), and gives no information on the rate of convergence. The results presented here give further information in two ways: First we give an algebraic rate of convergence and second we prove convergence in the L^1 norm, and hence the L^2 norm as well. Moreover, since $\|v\|_\infty^2 \leqslant 2\,\|v\|_2\,\|v'\|_2$ and we also establish an *a priori* bound on $\|v'\|_2$, see Section 2, our results also imply algebraic convergence in the L^∞ norm.

While we know of no other work on this problem for (1.12), there is a recent work on the analogous problem for a closely related equation. For the Cahn–Hilliard equation Bricmont, Kupianen, and Taskinen,[4] used renormalization group methods to prove algebraic convergence of small perturbations of the fronts and characterized the front to which one converges with the condition corresponding to (1.14). The relation between this equation and the Cahn–Hilliard equation is discussed at the end of the introduction. First however, we describe our approach to the problem at hand.

Naturally, the dissipation inequality (1.7) plays a fundamental role in the proof of Theorem 1.1. However, there are significant obstacles in the way of exploiting this dissipation due to other *non-dissipative and non-monotone features of the evolution.* These non dissipative features, ultimately due to the non-locality of our equation, prevent applicability of standard methods for estimating decay rates, and force the development of new methods for the solution of the problem.

This can be understood by looking at the linearized equation. The following paragraphs contain the definitions of some operators that appear in the linearized equation, but which are also fundamental in the investigation of the full equation. The most important of these is the second variation $\mathscr{A}_a$ of the free energy at a front $\bar{m}_a$, which is given by

$$\langle u, \mathscr{A}_a u \rangle_{L^2} = \frac{d^2}{ds^2} \mathscr{F}(\bar{m}_a + su)\Big|_{s=0} \tag{1.20}$$

Since the free energy is locally minimized at any front $\bar{m}_a$, each $\mathscr{A}_a$ is a non-negative operator on L^2. One easily works out that

$$\mathscr{A}_a u = \frac{1}{\beta(1-\bar{m}_a^2)} u - J * u \tag{1.21}$$

Moreover, since $\mathscr{F}$ is translation invariant, each of the fronts $\bar{m}_a$ satisfies the Euler equation

$$\frac{\delta \mathscr{F}}{\delta m}(\bar{m}_a) = \operatorname{arctanh}(\bar{m}_a) - \beta J * \bar{m}_a = 0 \tag{1.22}$$

Differentiating this with respect to a at $a=0$ we obtain:

$$\mathscr{A}_a \bar{m}'_a = 0 \tag{1.23}$$

and hence $\bar{m}'_a$ is in the null space of $\mathscr{A}_a$. In fact, following ref. 13, where a closely related operator is treated, it can be proven that $\bar{m}'_a$ in fact spans the null space of $\mathscr{A}_a$ and that there is an $\alpha > 0$ such that

$$\langle u, \mathscr{A}_a u \rangle_{L^2} \geqslant \alpha \|u\|_2^2 \qquad \text{for all } u \text{ with } \langle u, \bar{m}'_a \rangle_{L^2} = 0 \tag{1.24}$$

It is now easy to do the linearization about $\bar{m}_{a(t)}$. Define $v(t)$ by

$$v(t) = m(t) - \bar{m}_{a(t)} \tag{1.25}$$

By the definition of $a(t)$ as the value of a that minimizes $\|m(t) - \bar{m}_a\|_2^2$, one has

$$\langle v, \bar{m}'_{a(t)} \rangle_{L^2} = 0 \tag{1.26}$$

which shall play a crucial role in what follows.

To simplify the notation, we will drop the subscripted $a(t)$ whenever a particular solution $m(t)$ is under consideration, and shall write $\bar{m}$ in place of $\bar{m}_{a(t)}$ and $\mathscr{A}$ in place of $\mathscr{A}_{a(t)}$. That is, we maintain the convention that

$\bar{m}$ always denotes the closest front to $m(t)$ in L^2, and $\mathscr{A}$ always denotes the second variation of the free energy at this front.

Now noting that

$$\frac{\partial}{\partial t} v = \frac{\partial}{\partial t} m - \frac{\partial}{\partial t} \bar{m}$$

and inserting

$$\frac{\delta \mathscr{F}}{\delta m} (\bar{m} + v) = \mathscr{A} v + O(v^2) \tag{1.27}$$

into (1.12) we obtain the linearized equation:

$$\frac{\partial v}{\partial t} = \frac{\partial}{\partial x} \left(\sigma(\bar{m}) \frac{\partial}{\partial x} \mathscr{A} v \right) + \dot{a}(t) \bar{m}' \tag{1.28}$$

where $\dot{a}(t)$ denotes the time derivative of $a(t)$, and, as always, σ the mobility function (1.6).

Since $\mathscr{F}(\bar{m} + v) = \frac{1}{2} \langle v, \mathscr{A} v \rangle_{L^2} + O(v^3)$, the analog of (1.7) for the linearized equation (1.28) is

$$\frac{d}{dt} \langle v, \mathscr{A} v \rangle_{L^2} = -2 \int \sigma(\bar{m}) \left[\frac{\partial}{\partial x} \mathscr{A} v \right]^2 dx \tag{1.29}$$

Notice that by (1.23), $\dot{a}(t)$ doesn't appear in (1.29).

Because of the derivatives, the operator in (1.28) has no spectral gap. However this would be no problem if we had *a priori* control of the size L^1 norm of a solution. Suppose, for example, it was established that

$$\sup_{t > 0} \|v(t)\|_1 \leqslant c_0 < \infty \tag{1.30}$$

Then applying the Nash inequality

$$\|\psi'\|_2^2 \geqslant K \|\psi\|_2^6 / \|\psi\|_1^4 \tag{1.31}$$

one would have

$$\frac{d}{dt} \|\mathscr{A}^{1/2} v\|_2^2 \leqslant -2K\beta(1 - m_\beta^2) \|\mathscr{A} v\|_2^6 / \|\mathscr{A} v\|_1^4$$

Now $\mathscr{A}$ is clearly bounded on L^1 so that (1.30) would imply

$$\sup_{t > 0} \|\mathscr{A} v(t)\|_1 \leqslant c_1 < \infty$$

Also, since v is orthogonal to $\bar{m}'$ as noted above (1.26), one has from (1.24) that

$$\|\mathscr{A} v\|_2^2 \geqslant \alpha \, \|\mathscr{A}^{1/2} v\|_2^2$$

so that finally we would have

$$\frac{d}{dt} \|\mathscr{A}^{1/2} v\|_2^2 \leqslant -\frac{2K\alpha^3}{(c_1)^4} \beta(1-m_\beta^2) \, \|\mathscr{A}^{1/2} v\|_2^6$$

This differential inequality for $\|\mathscr{A}^{1/2} v\|_2^2$ would then imply that it decays away like $t^{-1/2}$.

This approach is standard in the study of parabolic equations

$$\frac{\partial u}{\partial t} = \nabla \cdot (a \, \nabla u) \tag{1.32}$$

and it works because for solutions of (1.32),

$$\frac{d}{dt} \int |u(x, t)| \, dx \leqslant 0 \tag{1.33}$$

so that the analog of (1.30) trivially holds.

However, for neither the full equation nor the linearized equation is the analog of (1.33) true. In fact, the rate of increase of the L^1 norm of a perturbation of a front can be arbitrarily large.

To see this, first we deduce the full non-linear evolution for v by inserting $m = \bar{m} + v$ into (1.12), and using (1.23) to eliminate the terms that are inhomogeneous in v. The result is:

$$\begin{aligned} \frac{\partial v}{\partial t} &= (\sigma(\bar{m})(\mathscr{A} v)')' + \beta(v^2 \, J * \bar{m}')' \\ &\quad + \beta(v(v + 2\bar{m}) \, J * v')' + \dot{a}(t) \, \bar{m}' \end{aligned} \tag{1.34}$$

Since the right hand side is a total derivative,

$$\frac{d}{dt} \int_{\mathbb{R}} v(x, t) \, dx = 0$$

as one expects from the conservation law in the underlying stochastic dynamics. Things are not so simple with the L^1 norm. To see the problem in the simplest way, suppose that v is smooth, and that at some time t, $v(x, t) = 0$ exactly at $x = 0$. Then

$$\frac{d}{dt} \int_{\mathbb{R}} |v(x, t)| \, dx = \int_{\mathbb{R}} \operatorname{sgn}(v(x, t)) \frac{d}{dt} v(x, t) \, dx$$
$$= -|v'(0, t)| + \beta(1 - \bar{m}^2(0, t))(J * v'(0, t))$$

If for example $v(x, t) = cx^3$, $c > 0$, on a neighborhood of 0 that is twice as wide as the support of J, one has

$$\frac{d}{dt} \int_{\mathbb{R}} |v(x, t)| \, dx \geqslant \beta(1 - m_\beta^2) \, 3c \int x^2 J(x) \, dx$$

One can easily construct examples, using more zeros, for which the rate of increase of the L^1 norm is an arbitrarily large multiple of the L^2 norm. This makes quite difficult to establish (1.30).

Moreover, there are other problematic non-dissipative features: The non-locality prevents anything except a small vestige of the maximum principle from applying. Indeed, suppose that m is a solution of (1.12), and that x_0 is some value of x for which $m(x_0) = \|m\|_\infty$. Then, with primes denoting derivatives with respect to x,

$$\frac{\partial}{\partial t} m(x_0, t) = m''(x_0, t) - \beta(1 - m^2(x_0, t))[J * m''(x_0, t)] \tag{1.35}$$

which can have either sign unless $(1 - m^2(x_0, t)) = 0$, in which case it is non-positive. Thus, solutions of (1.12) with initial data m_0 satisfying $\|m_0\|_\infty \leqslant 1$, will have $\|m(\cdot, t)\|_\infty \leqslant 1$ for all t, but even if $\|m_0\|_\infty < 1$, there is nothing to prevent $\|m(\cdot, t)\|_\infty = 1$ from occurring at some later time t.

This is a significant difficulty since the formal Frechet derivative of the free energy is

$$\frac{\delta \mathscr{F}}{\delta m} = \frac{1}{\beta} \operatorname{arctanh}(m) - J * m$$

Now, for any m with $-1 \leqslant m \leqslant 1$, $J * m$ is bounded, but $\operatorname{arctanh}(m) = \pm\infty$ on $\{x \mid m(x) = \pm 1\}$. Thus the free energy is *not Frechet differentiable* on the natural set of functions that is invariant under the evolution prescribed by (1.12). This means that some care must be taken with the use of the key dissipativity property (1.7) whose formal derivation depends on this Frechet differentiability. Even worse, however, is that the mobility (1.6) vanishes where $m = \pm 1$, and with it the local contribution to the dissipation in (1.7).

Having described the features that prevent applicability of standard methods, we now describe the strategy that is developed here. Clearly, we would like to have a substitute for the L^1 norm whose possible rate of growth is more readily estimated. The role of the L^1 norm in the Nash inequality (1.31) is to control the "tails" of ψ and to ensure that these decay strictly faster than they would have to for ψ to merely belong to L^2. Moments also measure tail decay, and their growth is relatively easy to estimate for our evolution. Thus, a key part of our analysis will be to bound the rate of increase of

$$\int x^2 \, |v(x,t)|^2 \, dx$$

or actually a closely related quantity as we shall explain. These estimates will then be used together with the inequality commonly known as the "uncertainty principle" i.e.,

$$\left(\int x^2 \, |\psi(x)|^2 \, dx\right)\left(\int |\psi'(x)|^2 \, dx\right) \geqslant \frac{1}{4}\left(\int |\psi(x)|^2 \, dx\right)^2 \tag{1.36}$$

The inequality (1.36) is Theorem 226 in ref. 18 where it is attributed to Herman Weyl.[22]

To illustrate this method for obtaining decay rates, consider a solution $u(x,t)$ of the heat equation

$$\frac{\partial}{\partial t} u(x,t) = u''(x,t)$$

Define

$$f(t) = \int |u(x,t)|^2 \, dx \qquad \text{and} \qquad \phi(t) = \int x^2 \, |u(x,t)|^2 \, dx$$

Then, one easily finds that

$$\begin{aligned} \frac{d}{dt}\phi(t) &= 2\int x^2 u(x,t)\, u''(x,t)\, dx \\ &= -4\int u(x,t)\, x u'(x,t)\, dx - 2\int |x u'(x,t)|^2 \, dx \\ &= -2\int |x u'(x,t) + u(x,t)|^2 \, dx + 2\int |u(x,t)|^2 \, dx \\ &\leqslant \frac{3}{2} f(t) \end{aligned} \tag{1.37}$$

(Here we have used the fact that the operator $x(d/dx)+\frac{1}{2}$ is skew-adjoint to obtain $\int |xu'(x,t)+u(x,t)|^2\,dx \geqslant \|u\|_2^2/4$.) Next, applying the uncertainty principle (1.36),

$$\begin{aligned}\frac{d}{dt}f(t) &= -2\int |u'(x,t)|^2\,dx\\ &\leqslant -\frac{1}{2}\left(\int |u'(x,t)|^2\,dx\right)^2\left(\int |xu(x,t)|^2\,dx\right)^{-1}\\ &= -\frac{1}{2}\frac{f^2(t)}{\phi(t)}\end{aligned}$$

Therefore we have a system of differential inequalities

$$\begin{aligned}\frac{d}{dt}f(t) &\leqslant -A\frac{f(t)^2}{\phi(t)}\\ \frac{d}{dt}\phi(t) &\leqslant Bf(t)\end{aligned} \tag{1.38}$$

We prove in Section 5 that any solution of the system of differential inequalities (1.38) satisfies the bounds

$$\begin{aligned}f(t) &\leqslant f(0)^{1-q}\,\phi(0)^q\left(\frac{\phi(0)}{f(0)}+(A+B)\,t\right)^{-q}\\ \phi(t) &\leqslant f(0)^{1-q}\,\phi(0)^q\left(\frac{\phi(0)}{f(0)}+(A+B)\,t\right)^{1-q}\end{aligned}$$

where

$$q=\frac{A}{A+B}$$

If we apply this to the heat equation, we get a decay rate like $t^{-1/4}$.

To apply this to (1.12) we take

$$f(t)=\mathscr{F}(\bar{m}+v(t))-\mathscr{F}(\bar{m}) \qquad \text{and} \qquad \phi(t)=\int \sigma(\bar{m})\,x^2\,|\mathscr{A}v(x,t)|^2\,dx \tag{1.39}$$

where, as always, σ is the mobility function given in (1.6).

For these choices of f and ϕ, we prove in Section 3 that there is a finite constant A so that the first inequality of (1.38) holds. A key step in Section 3 is to prove that the dissipation rate for the excess free energy is essentially bounded below by the dissipation for the linearized equation (1.29). This entails delicate lower bounds on $\|(\mathscr{A}v)'\|_2^2$. The bound obtained, presented in Theorem 3.2, is one of the main results of this paper.

In Section 4 we show that there is a finite constant B such that the second inequality holds, for all t such that $|a(t)| \leqslant 1$. As explained earlier, we shall only prove the L^1 estimates we need to establish that this does indeed hold for all t for sufficiently small initial perturbations of a front in 6. However, there is one important special case for which the estimates obtained here already suffice: namely antisymmetric initial data. In that case, $a(t)=0$ for all times t. The arguments in Sections 3 and 4 depend heavily on properties of the free energy functional and smoothing properties of (1.12). These are presented in Section 2. Finally in Section 5 we prove the theorem on solutions of (1.38), Theorem 5.1, and the following result covering antisymmetric initial data:

Theorem 1.3. Consider initial data $m_0(x)$ for (1.12) such that

$$\int x^2(m_0(x)-\bar{m}_0(x))^2\,dx \leqslant c_0$$

where c_0 is any positive constant. Then there are constants $C<\infty$, $\varepsilon>0$ and $q>0$ depending only on c_0, β and J such that for all inital data m_0 with $-1 \leqslant m_0 \leqslant 1$, and with

$$\int (m_0(x)-\bar{m}_0(x))^2\,dx \leqslant \varepsilon$$

the excess free energy $\mathscr{F}(m(t))-\mathscr{F}(m_0)$ of the corresponding solution $m(t)$ of (1.12) satisfies

$$\mathscr{F}(m(t))-\mathscr{F}(\bar{m}) \leqslant C(1+t)^{-q}$$

provided $\sup_{t>0}|a(t)| \leqslant 1$, which is the case when m_0 is antisymmetric.

We close the introduction by breafly discussing some related problems.

If one replaces the Kawasaki dynamics considered here by Glauber dynamics with an appropriate choice of transition rates, the time evolution for the local magnetization density that one obtains is

$$\frac{\partial m(x,t)}{\partial t} = \tanh(\beta J * m)(x,t) - m(x,t) \tag{1.40}$$

which decreases the same free energy functional, though it does not conserve the magnetization. Existence, unicity, asymptotic exponential stability for (1.40) have been proved in refs. 13 and 14. Both the Glauber and the Kawasaki evolutions drive down the same excess free energy functional, and hence, both have the same front profiles. However, the relaxation to these fronts is much faster in the Glauber case. Since under Kawasaki dynamics the total magnetization is conserved, a local excess of magnetization can only be relaxed by transporting it to an interface or to infinity. This can take an arbitrarily long time for perturbations that are small in L^2, but are nearly constant on large intervals far from the interface. The moment condition on the initial data in Theorem 1.1 is there on this account. While under Glauber dynamics, such perturbations would be quickly relaxed by local spin flips, in the case at hand, they must diffuse away. This is why the algebraic rates of relaxation are all that one can have, and why the time taken to relax depends on the localization of the initial perturbation, as controlled by the moment condition of Theorem 1.1.

The Equation (1.1) is closely related to the Cahn–Hilliard equation, which can also be written in the form (1.5) for a different free energy, namely

$$\tilde{\mathscr{F}}(m) = \int_{\mathbb{R}^n} (|\nabla m(x)|^2 + cF(m(x)))\, dx \tag{1.41}$$

where $F(m) = (m^2-1)^2/4$ and c is a constant. However, if we replace J in (1.1) by $J^{(\lambda)}$ where

$$J^{(\lambda)}(x) = \lambda^{-n} J(x/\lambda)$$

then for smooth m,

$$\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} J^{(\lambda)}(x-y)[m(x)-m(y)]^2\, dx\, dy = \lambda^2 \operatorname{var}(J) \int_{\mathbb{R}^n} |\nabla m(x)|^2\, dx + O(\lambda^4)$$

where $\operatorname{var}(J)$ denote the variance of J. Moreover, the general behavior of the potentials is the same: they are both simple double wells.

In fact, the relation is close enough that one expects the motion of the interfaces in the sharp interface scaling limit to be the same for (1.1) and the Cahn–Hilliard equation. Analogous results for (1.40) and motion by mean curvature have been obtained previously.[11, 19]

2. SMOOTHING ESTIMATES AND DIFFERENTIABILITY OF THE FREE ENERGY

In this section we quote some technical results upon which our analysis in the following sections depends. These results are proved in full detail in ref. 7, and here we sketch the proofs.

First, throughout the paper, we are concerned with the behavior of the free energy functional $\mathscr{F}$ on the set of profiles

$$\mathscr{M} = \{m \mid \|m\|_\infty \leqslant 1 \text{ and } \|m - \bar{m}\|_2 < \infty\} \tag{2.1}$$

A technical difficulty arises in that $\mathscr{F}$ is *not* Frechet differentiable on $\mathscr{M}$. As explained in the introduction, the problem arises on the boundaries where

$$\bar{m} + v = \pm 1$$

The formal Frechet derivative of $\mathscr{G}$ is

$$\frac{\delta \mathscr{G}}{\delta v} = \frac{1}{\beta} \operatorname{arctanh}(\bar{m} + v) - J * (\bar{m} + v) \tag{2.2}$$

The convolution term satisfies $\|J * (\bar{m} + v)\|_\infty \leqslant 1$, but on any set where $\bar{m} + v = \pm 1$ we have

$$\operatorname{arctanh}(\bar{m} + v) = \pm \infty$$

Because there is no maximum principle for our evolution equation for values of m other than ± 1, there is nothing to keep solutions from reaching these values even if the initial data m_0 satisfies $\|m_0\|_\infty < 1$. Therefore, some care is required even to show that the free energy function is a Lyapunov function for (1.12). Nonetheless, we have the following result:

Theorem 2.1. For all initial data m_0 in $\mathscr{M}$ with $\mathscr{F}(m_0) < \infty$, the corresponding solution $m(x, t)$ of equation (1.12) satisfies

$$\mathscr{F}(m(0)) = \mathscr{F}(m(t)) + \int_0^t \mathscr{I}(m(s))\, ds \qquad \text{for all} \quad t > 0 \tag{2.3}$$

where

$$\mathscr{I}(m) = \int \sigma(m) \left(\frac{\partial}{\partial x} \frac{\delta \mathscr{F}}{\delta m} \right)^2 dx \tag{2.4}$$

In particular, $\mathscr{F}(m(t))$ is monotonically decreasing.

The proof of this theorem, as well as most of the results established here, depends on certain smoothing properties of the evolution (1.12). The required *a priori* smoothing estimates are summarized in the following theorem:

Theorem 2.2. For any positive number κ, there is a $\delta_* > 0$ so that for all $\delta_0 < \delta_1$ with $\delta_1 \leqslant \delta_*$, there are strictly positive constants $\varepsilon(\delta_0, \delta_1, \kappa_1)$ and $t_0(\delta_0, \delta_1, \kappa_1)$, depending only on the indicated quantities, such that whenever $m(t)$ is a solution of (1.12) with initial data

$$m(x, 0) = \bar{m}_0(x) + v_0(x)$$

for which $\|v_0\|_2 \leqslant \varepsilon(\delta_0, \delta_1, \kappa_1)$, one has:

$$\|v(t_0)\|_2 \leqslant \delta_0$$

and

$$\|v'(t)\|_2 \leqslant \kappa_1$$

for all $t > t_0$ such that $\|v(t)\|_2 \leqslant \delta_1$. Moreover,

$$\lim_{\delta_1 \to 0} t_0(\delta_0, \delta_1, \kappa_1) = 0$$

and for all t such that $\|v(t)\|_2$ is finite, $v(t)$ is C^∞ in x, with all of its spatial derivatives square integrable.

Finally, in case

$$\int |xv_0(x)|^2 \, dx = c_0 < \infty$$

then, further by decreasing δ_0 if need be, we have that

$$\int |xv(x, t_0)|^2 \, dx \leqslant 2c_0$$

We briefly sketch the proof by applying the method upon which it is based to the heat equation, for which such a result is easy to prove. The method sketched here is easily adapted to (1.34), as proved in ref. 7.

For a solution u of the heat equation

$$\frac{\partial u}{\partial t} = u''$$

one has

$$\frac{d}{dt} \|u'(t)\|_2^2 = -2 \, \|u''(t)\|_2^2$$

Then since

$$\|u'\|_2^2 = -\int u''(x)\, u(x)\, dx \leqslant \|u''\|_2\, \|u\|_2 \tag{2.5}$$

one has

$$\frac{d}{dt}\|u'(t)\|_2^2 \leqslant -2\,\frac{\|u'(t)\|_2^4}{\|u(t)\|_2^2}$$

Define $y(t) = 1/\|u(t)\|_2^2$. Then on any interval $[0, T_0]$ on which $\|u(t)\|_2^2 \leqslant \delta$, one has

$$\frac{d}{dt}\, y(t) \geqslant \frac{1}{\delta}$$

and hence

$$y(t) \geqslant y(0) + \frac{t}{\delta}$$

Thus, even if $\|u'(0)\|_2 = \infty$, $\|u'(t)\|_2^2 \leqslant \delta/t$.

If one does the same for our equation (1.34), one obtains a bound of the form

$$\frac{d}{dt}\|v'(t)_2^2 \leqslant -\|v''(t)\|_2^2 + C(\|v(t)\|_2^2 + \|v(t)\|_2^5)$$

where C is a constant depending only on β and J. From this, one derives a differential inequality for $\|v'(t)\|_2^2$ which allows one to estimate the time t_0 it takes for $\|v'(t)\|_2^2$ to decrease to κ_1 under the assumption that

$$\|v(t)\|_2^2 \leqslant \delta \tag{2.6}$$

for some sufficiently small—depending on κ_1—value of δ. Now one just needs to know that (2.6) holds on a time interval long enough for the desired decrease to occur. Of course for the heat equation, if (2.6) holds initially, it does so for all time. In our case, the following lemma shows that there is an $\varepsilon > 0$ such that if $\|v(0)\|_2^2 < \varepsilon$, then (2.6) holds on $[0, 2t_0]$.

Once one has control on the first derivative, higher order derivatives are controlled by iteration and induction. For the details, see ref. 7. Next lemma is established by deriving a differential inequality for $\|v(t)\|_2^2$ from (1.34).

Lemma 2.3. Let v be a solution of (1.34). Then there are constants A and B depending only on β and J so that

$$\|v(t)\|_2^2 \leqslant 3\ \|v(0)\|_2^2$$

for all t such that

$$e^{At} \leqslant B/\|v(0)\|_2^2$$

Finally, since (1.34) contains a term involving $\dot{a}(t)$, one also needs estimates on this quantity. The following shall also be used in Section 4.

Theorem 2.4. Let m be a solution of (1.12). Then there is a $\delta_0 > 0$ such that whenever

$$\inf_{a \in \mathbb{R}} \{\|m(t) - \bar{m}_a\|_2\} < \delta_0 \tag{2.7}$$

there is a unique value $a(t)$ at which the infimum in (2.7) is attained. Moreover, for any $\kappa > 0$, there is a $\delta_1(\kappa, \beta, J)$ such that whenever $\|v'(t)\|_2 \leqslant \kappa$ and $\|v(t)\|_2 \leqslant \delta_1$, $a(t)$ is differentiable and

$$|\dot{a}(t)| \leqslant D(\kappa, \beta, J)\ \|v(t)\|_2 \tag{2.8}$$

where $D(\kappa, \beta, J)$ is a constant depending only on κ, β and J.

Proof. Let $a(t)$ be any minimizer in (2.7). Clearly there is at least one, and what we must show is that there is exactly one. Define $d(b) = \|m(t) - \bar{m}_b\|_2^2$. Taking two derivatives,

$$d''(b) = -2 \int m(x, t)\, \bar{m}_b''(x)\, dx = 2 \int \bar{m}_{a(t)}' \bar{m}_b'(x)\, dx - 2 \int v(x, t)\, \bar{m}_b''(x)\, dx$$

Hence,

$$d''(b) \geqslant 2 \int \bar{m}_{a(t)}' \bar{m}_b'(x)\, dx - 2\delta_0\ \|\bar{m}_b''\|_2$$

But by continuity, $\int \bar{m}_{a(t)}' \bar{m}_b'(x)\, dx > \|\bar{m}_b'\|_2^2/2$ on some interval $(a(t) - c, a(t) + c)$ for some c depending only on β and J. Therefore, choose

$$\delta_0 \leqslant \frac{\|\bar{m}_b'\|_2^2}{4\ \|\bar{m}_b''\|_2} \tag{2.9}$$

and it follows that $d''(b)>0$ on $(a(t)-c, a(t)+c)$, and hence there is exactly one critical point of $d(b)$ on $(a(t)-c, a(t)+c)$. However, if b is any value with

$$\|m(t)-\bar{m}_b\|_2=\|m(t)-\bar{m}_{a(t)}\|_2$$

it follows that

$$\|\bar{m}_b-\bar{m}_{a(t)}\|_2\leqslant 2\ \|m(t)-\bar{m}_{a(t)}\|_2\leqslant 2\delta_0$$

But there is a constant K depending only on β and J so that

$$\|\bar{m}_b-\bar{m}_a\|_2\geqslant\frac{(b-a)^2}{C+(b-a)^2}$$

and thus,

$$\frac{(b-a)^2}{C+(b-a)^2}\leqslant 2\delta_0$$

Decreasing δ_0 if necessary, one can ensure that $|b-a|<c$. Hence any putative second minimum must occur within $(a(t)-c, a(t)+c)$ where there is only the single critical point $a(t)$. Hence there is no other minimum. This proves that $a(t)$ is a well-defined function under the condition (2.7). We now establish a bound on its derivative. The starting point is the equation for the minimum:

$$\int (m(t)-\bar{m}_{a(t)})\ \bar{m}'_{a(t)}\ dx=0$$

which holds for all t. Differentiating in t, one obtains

$$\dot{a}(t)(\|\bar{m}'_a\|_2^2-\langle v, \bar{m}''_a\rangle_{L^2})=-\int\frac{\partial m}{\partial t}\,\bar{m}'_a$$

Thus, we have

$$|\dot{a}(t)|\leqslant 2\left|\int\sigma(m)\left(\frac{\delta\mathcal{F}}{\delta m}\right)'\ \bar{m}''_a\ dx\right|$$

$$2\left|\int(\sigma(m)\ \bar{m}''_a)'\ \frac{\delta\mathcal{F}}{\delta m}\ dx\right|$$

from which the result easily follows, since the L^2 norm of $\delta\mathcal{F}/\delta m(\bar{m}+v)$ is bounded by a constant times the L^2 norm of v whenever $\|v'\|_2\leqslant\kappa$ and $\|v\|_2\leqslant\delta_1$ with δ_1 sufficiently small to guarantee that $\|v\|_\infty\leqslant(1-m_\beta^2)/2$. ∎

3. DISSIPATION RATE OF THE FREE ENERGY

We know already from Theorem 2.1 that $\mathcal{F}(m(t))$ is monotonically decreasing along the solution of (1.12). In this section we establish a bound on the rate at which the excess free energy $\mathcal{F}(m(t)) - \mathcal{F}(\bar{m})$ is decreasing in term of itself. Essential ingredients are the "uncertainty principle" (1.36), the differentiability properties of the functional $\mathcal{F}(m)$ given in Section 2, and a lower bound on $\mathcal{I}(\bar{m}+v)$ in terms of a quadratic form in v. This lower bound, given in Theorem 3.2 below, is the main result of this section, and one of the main results of the paper. The quadratic form in question comes from the dissipativity of the linearized equation (1.29), and while it is natural to expect such a result in a sufficiently small neighborhood of a front, the proof is we present is quite intricate. The fact that $\mathcal{A}$ has a non-trivial null space greatly complicates the control of the error terms in the linearization. In any case, using the above ingredients, we shall prove:

Theorem 3.1. Let $m(\cdot, t)$ be a solution of (1.12). Then there are $\delta_0 > 0$ and $\kappa > 0$ so that at all times t for which $\|v'(t)\|_2 < \kappa$ and $\|v(t)\|_2 < \delta_0$ we have that

$$\frac{d}{dt}[\mathcal{F}(mt)) - \mathcal{F}(\bar{m})] \leqslant -A \frac{[\mathcal{F}(m(t)) - \mathcal{F}(\bar{m})]^2}{\phi(t)} \tag{3.1}$$

where A and δ_0 depend on β, J, κ.

From Theorem 2.1 we know that the excess free energy is dissipated with rate

$$\mathcal{I}(m) = \int \sigma(m(x)) \left[\frac{\partial}{\partial x} \left(\frac{1}{\beta} \operatorname{arctanh} m(x) - (J * m)(x) \right) \right]^2 dx \tag{3.2}$$

Most of the work in proving Theorem 3.1 goes into showing that in a sufficiently small neighborhood of a front the dissipation of the excess free energy functional is essentially bounded below by the dissipation for the linearized equation (1.29).

Theorem 3.2. Set $v = m - \bar{m}$ where $\bar{m}$ is the closest front in L^2 to m. Then for any $\varepsilon > 0$ small enough there are $\delta(\beta, J, \varepsilon) > 0$ and $\kappa(\beta, J, \varepsilon) > 0$ such that for $\|v'\|_2 \leqslant \kappa(\beta, J, \varepsilon)$, $\|v\|_2 \leqslant \delta(\beta, J, \varepsilon)$

$$\mathcal{I}(m) \geqslant (1 - 3\varepsilon) \int \sigma(\bar{m}(x))[(\mathcal{A}v)'(x)]^2 \, dx \tag{3.3}$$

where $\mathcal{A}$ is the linear operator defined in (1.21).

The proof of Theorem 3.2 depends on several steps. We start stating and proving two lemmas.

Lemma 3.3. Let $m=\bar{m}+v$ and the mobility $\sigma(m)$ defined in (1.6). For any $\kappa>0$ and $\varepsilon>0$ there exists $\delta_1(\kappa, \varepsilon, \beta, J)>0$ such that

$$(1-\varepsilon)\,\sigma(\bar{m})\leqslant\sigma(m)\leqslant\sigma(\bar{m})(1+\varepsilon) \tag{3.4}$$

provided $\|v'\|_2\leqslant\kappa$ and $\|v\|_2\leqslant\delta_1$.

Proof. Since $\sigma(m)=\beta(1-m^2)$, $\sigma(m)=\sigma(\bar{m})[1+(1/\sigma(\bar{m}))\,\beta(2\bar{m}+v)\,v]$. One easily obtains the pointwise bound

$$\left|\frac{1}{\sigma(\bar{m})}\,\beta(2\bar{m}+v)\,v\right|\leqslant c_\beta\,\|v\|_2^{1/2}\,\kappa^{1/2}[1+\|v\|_2^{1/2}\,\kappa^{1/2}]\leqslant\varepsilon$$

provided $\delta_1\leqslant c_\beta\varepsilon^2/\kappa$ and $\|v\|_2\leqslant\delta_1$, where we denote by c_β a constant changing line to line depending only on β and J. ∎

Lemma 3.4. Let $v\in L^2(\mathbb{R})$, $v'\in L^2(\mathbb{R})$ and $\int v(x)\,\bar{m}'(x)\,dx=0$ then there exists a positive constant γ, depending on β and J, such that

$$\int[(\mathscr{A}v)']^2\,dx\geqslant\gamma\,\|Pv'\|_2^2 \tag{3.5}$$

where $\mathscr{A}$ is the linear operator defined in (1.21) and P is the orthogonal projection on the orthogonal complement of $\bar{m}''$.

Proof. We may write

$$v(x)=v(y)+\int_y^x v'(z)\,dz$$

We then multiply both terms by $\bar{m}'(y)$, since $\int v(y)\,\bar{m}'(y)\,dy=0$ we have

$$v(x)=\frac{1}{2m_\beta}\int_{-\infty}^{\infty}\bar{m}'(y)\left(\int_y^x v'(z)\right)dy \tag{3.6}$$

Hence

$$(\mathscr{A}v)'=\mathscr{A}v'+Kv'$$

where

$$K\phi(x)=\frac{1}{\beta}\,\frac{\bar{m}\bar{m}'}{(1-\bar{m}^2)^2}\,\frac{1}{m_\beta}\int_{-\infty}^{\infty}\bar{m}'(y)\left(\int_y^x\phi(z)\,dz\right)dy$$

The operator K is compact on L^2. In fact, it is easy to verify that

(i) for all $\varepsilon > 0$ there is $h_\varepsilon > 0$ such that for all $0 \leqslant h < h_\varepsilon$

$$\int |K\phi(x+h) - K\phi(x)|^2 \, dx \leqslant \varepsilon$$

(ii) for all $\varepsilon > 0$ there is $X_\varepsilon > 0$ such that for all $X \geqslant X_\varepsilon$

$$\int_{|x| \geqslant X} |K\phi(x)|^2 \, dx \leqslant \varepsilon$$

Property (i) follows from the continuity of integral.
Moreover from (3.6) we have that,

$$\begin{aligned} |v(x)| &\leqslant \frac{1}{2m_\beta} \left(\int \bar{m}'(y) \, |x-y|^{1/2} \, dy \right) \|v'\|_2 \\ &\leqslant \left(\frac{1}{2m_\beta} \int \bar{m}'(y) \, |x-y|^2 \, dy \right)^{1/4} \|v'\|_2 \\ &\leqslant C(1+|x|) \, \|v'\|_2 \end{aligned} \tag{3.7}$$

where $C > 0$ is a constant depending on β and J.
Then

$$\int |K\phi(x)|^2 \, dx \leqslant C \, \|\phi\|_2^2 \int (\bar{m}(x))^2 \, (1+|x|) \, dx \tag{3.8}$$

Then the (ii) statement follows from the rapid decay of $\bar{m}'$, see (1.10).
Using this notation and denoting by $*$ the adjoint operator in L^2, we can rewrite

$$\int |(\mathscr{A}v)'|^2 \, dx = \int v'(\mathscr{A}^2 + K^*\mathscr{A} + \mathscr{A}K + K^*K) \, v' \, dx$$

The operator $K^*\mathscr{A} + \mathscr{A}K + K^*K$ is compact since $\mathscr{A}$ is bounded and K is compact. Hence, by Weyl's theorem, the essential spectrum of $\mathscr{A}^2$ and $\mathscr{A}^2 + (K^*\mathscr{A} + \mathscr{A}K + K^*K)$ concide. By the above, both operators are non-negative, with zero being a simple eigenvalue. Moreover, it is clear from the above that the nullspace of $\mathscr{A}^2 + (K^*\mathscr{A} + \mathscr{A}K + K^*K)$ is spanned by $\bar{m}''$. Since the essential spectrum of $\mathscr{A}^2$ is strictly positive, it follows that

$\mathscr{A}^2 + (K^*\mathscr{A} + \mathscr{A}K + K^*K)$ is strictly positive of the orthogonal complement of $\bar{m}''$; i.e., for some $\gamma > 0$, depending on β and J.

$$\int v'(\mathscr{A}^2 + K^*\mathscr{A} + \mathscr{A}K + K^*K)\, v'\, dx \geqslant \gamma \int |Pv'|^2\, dx$$

where P is the orthogonal projection on the orthogonal complement of $\bar{m}''$. ∎

Proof of Theorem 3.2. To begin the proof we write

$$\frac{1}{\beta}\frac{m'}{1-m^2} - J * m' = \frac{1}{\beta}\left(\frac{m'}{1-m^2} - \frac{m'}{1-\bar{m}^2}\right) + \left(\frac{1}{\beta}\frac{m'}{1-\bar{m}^2} - J * m'\right)$$
$$= \frac{1}{\beta}\left(\frac{1}{1-m^2} - \frac{1}{1-\bar{m}^2}\right) m' + \mathscr{A}m'$$

Then since $\mathscr{A}\bar{m}' = 0$, $\mathscr{A}m' = \mathscr{A}v'$. Also,

$$\left(\frac{1}{1-m^2} - \frac{1}{1-\bar{m}^2}\right) = \frac{2\bar{m}}{(1-\bar{m}^2)^2}\, v + \frac{1 + 3\bar{m}^2 + 2\bar{m}v}{(1-\bar{m}^2)^2\,(1-m^2)}\, v^2$$

Hence we can rewrite

$$\frac{1}{\beta}\frac{m'}{1-m^2} - J * m' = \mathscr{A}v' + \frac{1}{\beta}\frac{2\bar{m}\bar{m}'}{(1-\bar{m}^2)^2}\, v + U(v)$$

where

$$U(v) = \frac{1}{\beta}\,\bar{m}'\,\frac{1 + 3\bar{m}^2 + 2\bar{m}v}{(1-\bar{m}^2)^2(1-m^2)}\, v^2 + \frac{1}{\beta}\frac{2\bar{m}}{(1-\bar{m}^2)^2}\, vv'$$
$$+ \frac{1}{\beta}\frac{1 + 3\bar{m}^2 + 2\bar{m}v}{(1-\bar{m}^2)^2(1-m^2)}\, v^2 v' \qquad (3.9)$$

But since

$$\frac{2\bar{m}\bar{m}'}{(1-\bar{m}^2)^2} = \frac{d}{dx}\frac{1}{(1-\bar{m}^2)}$$

this is the same as

$$\frac{1}{\beta}\frac{m'}{1-m^2} - J * m' = (\mathscr{A}v)' + U(v)$$

Therefore we have that

$$\mathscr{I}(m) = \int \sigma(m)[(\mathscr{A}v)' + U(v)]^2\, dx \tag{3.10}$$

Applying Lemma 3.3 we have that

$$\mathscr{I}(m) \geqslant (1-\varepsilon) \int \sigma(\bar{m})[(\mathscr{A}v)' + U(v)]^2\, dx \tag{3.11}$$

provided δ is less than the δ_1 of Lemma 3.3.

Now for any f and g in L^2 and for any number λ with $0 < \lambda < 1$,

$$\begin{aligned} \|f+g\|_2^2 &\geqslant \|f\|_2^2 + \|g\|_2^2 - 2\,\|f\|_2\,\|g\|_2 \\ &= \lambda\,\|f\|_2^2 + ((1-\lambda)\,\|f\|_2^2 + \|g\|_2^2 - 2\,\|f\|_2\,\|g\|_2) \\ &\geqslant \lambda\,\|f\|_2^2 - \left(\frac{1}{1-\lambda} - 1\right)\|g\|_2^2 \end{aligned} \tag{3.12}$$

We apply this in (3.11), with $\lambda = 1-\varepsilon$, where $\varepsilon > 0$ is small, arbitrarily chosen, obtaining

$$\begin{aligned} \int \sigma(\bar{m})[(\mathscr{A}v)' + U(v)]^2\, dx &\geqslant (1-2\varepsilon) \int \sigma(\bar{m})[(\mathscr{A}v)']^2\, dx \\ &\quad + \left[\varepsilon \int \sigma(\bar{m})[(\mathscr{A}v)']^2\, dx - \frac{1}{\varepsilon} \int \sigma(\bar{m})[U(v)]^2\, dx\right] \end{aligned} \tag{3.13}$$

The statement of theorem follows once it is shown that the last term in (3.13) is positive. Applying Lemma 3.4 to the last term in (3.13) one obtains

$$\begin{aligned} &\varepsilon \int \sigma(\bar{m})[(\mathscr{A}v)']^2\, dx - \frac{1}{\varepsilon} \int \sigma(\bar{m})[U(v)]^2\, dx \\ &\qquad \geqslant \varepsilon\gamma\sigma(m_\beta)\,\|Pv'\|_2^2 - \frac{1}{\varepsilon} \int \sigma(\bar{m})[U(v)]^2\, dx \end{aligned} \tag{3.14}$$

Lemma 3.5 and Lemma 3.6 below show that the last term in (3.14) is positive. Therefore the constant $\kappa(\beta, J, \varepsilon)$ in the statement of theorem has to be taken equal to that one of Lemma 3.6, since this is the only place where smallness condition is required for $\|v'\|_2$. The $\delta(\beta, J, \varepsilon)$ in the statement of the theorem has to be taken so that

$$\delta(\beta, J, \varepsilon) \leqslant \min\{\delta_1(\beta, J, \varepsilon), \delta_2(\beta, J, \varepsilon)\}$$

where $\delta_1(\beta, J, \varepsilon)$ is the δ_1 of Lemma 3.3 and $\delta_2(\beta, J, \varepsilon)$ is from Lemma 3.6 below. Accepting Lemmas 3.5 and 3.6 below, the theorem is proved. ∎

The work that remains to be done in Lemmas 3.5 and 3.6 is to compare $\|Pv'\|_2^2$ with $\|U(v)\|_2^2$.

It is not hard to estimate $\|U(v)\|_2^2$ in terms of $\|v'\|_2^3$, however the ratio $\|Pv'\|_2\,(\|v'\|_2)^{-1}$ can be arbitrarily small, so that it is not clear that an $O(\|v'\|_2^3)$ term is negligeable with respect to $O(\|Pv'\|_2^2)$. To see the situation more clearly, write

$$v' = \alpha\bar{m}'' + w' \tag{3.15}$$

where $\int w'm''\,dx = 0$ so that $Pv' = w'$. Note that, as indicated in our notation, Pv' is a derivative since v' and $\bar{m}''$ are derivatives. Hence, upon integration

$$v = \alpha\bar{m}' + w \tag{3.16}$$

The fact that $\langle v, \bar{m}'\rangle = 0$ means that $\|w\|_2$ cannot be too small. But what we need to know is that $\|w'\|_2 = \|Pv'\|_2$ is not too small. In general, these are simply two different things.

What provides the crucial connection here is that $(2m_\beta)^{-1}\,\bar{m}'(x)\,dx$ is a probability measure, so that

$$\langle v, \bar{m}'\rangle_{L^2} = 0$$

implies that

$$\|w\|_\infty \geqslant \frac{|\alpha|}{2m_\beta}\,\|\bar{m}'\|_2$$

Then one may use $\|w\|_\infty^2 \leqslant 2\|w'\|_2\|w\|_2$ to conclude that

$$|\alpha|^2 \leqslant \frac{8m_\beta^2}{\|\bar{m}'\|_2^2}\,\|w\|_2\,\|w'\|_2$$

Now suppose that $\|w'\|_2 \ll \|v'\|_2$—for otherwise there is no problem. Then $|\alpha|$ is comparable to $\|v'\|_2$ and one has

$$\|v'\|_2^4 \leqslant C(\beta, J)\,\|w'\|_2^2\,\|w\|_2^2$$

Hence $\|v'\|_2^4$ will be negligible with respect to $\|w'\|_2^2 = \|Pv'\|_2^2$ provided $\|w\|_2$ is sufficiently small. This in turn will be enforced by our conditions $\|v'\|_2 \leqslant \kappa$ and $\|v\|_2 \leqslant \delta$.

Hence, what we shall show in Lemma 3.5 is that in a sufficiently small neighborhood of a front, $\int \sigma(m)[U(v)]^2\,dx$ can be bounded by an arbitrarily small piece of $\|w'\|_2^2$ and a multiple of $\|v'\|_2^4$, or else simply by a small piece of $\|v'\|_2^2$. Then in Lemma 3.6, we use the ideas sketched above to obtain a lower bound for $\|Pv'\|_2^2$ in terms of either $\|v'\|_2^4$, or simply $\|v'\|_2^2$, which shows that in a sufficiently small neighborhood of a front, the difference in (3.14) is positive.

Lemma 3.5. Let $v \in L^2(\mathbb{R})$, $v' \in L^2(\mathbb{R})$. For any $\kappa > 0$ and $\varepsilon_0 > 0$ small enough, there exists $\delta(\kappa, \varepsilon_0, \beta, J) > 0$ such that for the non linear operator $U(v)$ defined in (3.9) the following two estimates hold

$$\int \sigma(m)[U(v)]^2\,dx \leqslant \varepsilon_0 \int |v'|^2\,dx \tag{3.17}$$

$$\int \sigma(m)[U(v)]^2\,dx \leqslant c(\varepsilon_0, \beta, J)\,\|v'\|_2^4 + \varepsilon_0\,\|w'\|_2^2 \tag{3.18}$$

provided $\|v'\|_2 \leqslant \kappa$, $\|v\|_2 \leqslant \delta$ and where $c(\varepsilon_0, \beta, J)$ is a positive constant.

Proof. Observe that for some constant C depending only on β and J

$$|U(v)|^2 \leqslant C(R(x)\,|v|^4 + |v|^2\,|v'|^2) \tag{3.19}$$

where $R(x)$ is non-negative and $\int R(x)\,dx = 1$. Then since $\|v\|_\infty^4 \leqslant 4\,\|v\|_2^2\,\|v'\|_2^2$,

$$\int R(x)\,|v|^4\,dx \leqslant 4\delta^2\,\|v'\|_2^2 \tag{3.20}$$

For the other term we have

$$\int |v|^2\,|v'|^2\,dx \leqslant 2\,\|v\|_2\,\|v'\|_2 \int |v'|^2\,dx \leqslant 2\delta\kappa \int |v'|^2\,dx \tag{3.21}$$

Hence we have

$$\int \sigma(m)[U(v)]^2\,dx \leqslant C(\delta, \kappa, \beta, J) \int |v'|^2\,dx$$

where $C(\delta, \kappa, \beta, J) = C\min\{2\delta\kappa; 4\delta^2\}$. Provided we take δ such that $C(\delta, \kappa, \beta, J) \leqslant \varepsilon_0$ we have (3.17).

We next derive (3.18). We insert the representation (3.6) for v in the first term of the right hand side on (3.19) obtaining, since $R(x)$ is rapidly decreasing, from properties (1.10),

$$\int R(x)\, v^4\, dx \leqslant \|v'\|_2^4 \int R(x)(1+|x|)^4\, dx \leqslant c(\beta, J)\, \|v'\|_2^4$$

where $c(\beta, J)$ is a positive constant.

For the other term in (3.19) we use for v the representation (3.16) obtaining

$$\int v^2(v')^2\, dx \leqslant 2 \int [\alpha^2(\bar{m}')^2 + w^2](v')^2\, dx \tag{3.22}$$

Moreover from (3.15)

$$|\alpha| \leqslant \|v'\|_2\, (\|\bar{m}''\|_2)^{-1} \tag{3.23}$$

and we obtain

$$\begin{aligned}\int v^2(v')^2\, dx &\leqslant 2\, \frac{\|v'\|_2^4}{\|\bar{m}''\|_2^2}\, \|\bar{m}'\|_2^2 + 4\, \|w\|_2\, \|w'\|_2\, \|v'\|_2^2 \\ &\leqslant 2\, \frac{\|v'\|_2^4}{\|\bar{m}''\|_2^2}\, \|\bar{m}'\|_2^2 + 16\lambda\, \|w\|_2^2\, \|w'\|_2^2 + \frac{1}{\lambda}\, \|v'\|_2^4 \\ &= \left(2\, \frac{\|\bar{m}'\|_2^2}{\|\bar{m}''\|_2^2} + \frac{1}{\lambda}\right) \|v'\|_2^4 + 16\lambda\, \|w\|_2^2\, \|w'\|_2^2\end{aligned} \tag{3.24}$$

for any chosen $\lambda > 0$. Because of (3.16) and (3.23)

$$\begin{aligned}\|w\|_2 = \|v - \alpha\bar{m}'\|_2 &\leqslant \|v\|_2 + |\alpha|\, \|\bar{m}'\|_2 \\ &\leqslant \|v\|_2 + \frac{\|v'\|_2}{\|\bar{m}''\|_2}\, \|\bar{m}'\|_2\end{aligned} \tag{3.25}$$

Applying (3.25) to the last term of (3.24) one obtains

$$\int v^2(v')^2\, dx \leqslant \left(2\, \frac{\|\bar{m}'\|_2^2}{\|\bar{m}''\|_2^2} + \frac{1}{\lambda}\right) \|v'\|_2^4 + 32\lambda \left[\, \|v\|_2^2 + \frac{\|v'\|_2^2}{\|\bar{m}''\|_2^2}\, \|\bar{m}'\|_2^2 \right] \|w'\|_2^2 \tag{3.26}$$

We can take λ such that $32\lambda[1 + (\kappa^2/(\|\bar{m}''\|_2^2))\, \|\bar{m}'\|_2^2] \leqslant \varepsilon_0$. Lemma is proved. ∎

Lemma 3.6. Let $v \in L^2(\mathbb{R})$ and $v' \in L^2(\mathbb{R})$. For any $\varepsilon > 0$ there exists $\delta_2(\varepsilon, \beta, J) > 0$ and $\kappa(\varepsilon, \beta, J)$ so that for $\|v'\|_2 \leqslant \kappa(\varepsilon, \beta, J)$, $\|v\|_2 \leqslant \delta_2(\varepsilon, \beta, J)$

$$\varepsilon\sigma(m_\beta)\,\gamma\,\|Pv'\|^2 - \frac{1}{\varepsilon}\int \sigma(\bar{m})[U(v)]^2\,dx \geqslant 0 \tag{3.27}$$

where γ is from Lemma 3.4 and $U(v)$ is defined in (3.9).

Proof. Suppose that

$$\|Pv'\|_2^2 > \tfrac{1}{2}\|v'\|_2^2 \tag{3.28}$$

Then from (3.17) of Lemma 3.5 we can take $\varepsilon_0 = \varepsilon^{2+r}$, with $r > 0$ obtaining

$$\frac{1}{2}\varepsilon\gamma\sigma(m_\beta)\,\|v'\|_2^2 - \frac{1}{\varepsilon}\varepsilon_0\,\|v'\|_2^2 = \varepsilon\,\|v'\|_2^2\left[\frac{1}{2}\gamma\sigma(m_\beta) - \varepsilon^r\right] \tag{3.29}$$

Choosing a suitable $r > 0$, the last term in (3.29) is positive. This implies that provided $\|v\|_2 \leqslant \delta(\varepsilon_0^{2+r}, \beta, J)$, where $\delta(\varepsilon_0^{2+r}, \beta, J)$ is from Lemma 3.5 (3.27) holds. Note that no further condition is required for $\|v'\|_2$ besides boundeness, i.e., the condition required to apply Lemma 3.5.

Next, suppose (3.28) is false, i.e.,

$$\|Pv'\|_2^2 \leqslant \tfrac{1}{2}\|v'\|_2^2 \tag{3.30}$$

then from (3.15)

$$\|v' - w'\|_2 \leqslant |\alpha|\,\|\bar{m}''\|_2$$

and applying (3.12) with $\lambda = \frac{1}{3}$ and (3.30) one obtains

$$|\alpha|^2\,\|\bar{m}''\|_2^2 \geqslant \|v' - w'\|_2^2 \geqslant \tfrac{1}{2}\|v'\|_2^2 - \tfrac{1}{2}\|w'\|_2^2 \geqslant \tfrac{1}{12}\|v'\|_2^2 \tag{3.31}$$

Therefore

$$\|v'\|_2^2 \leqslant 12\,\|\bar{m}''\|_2^2\,|\alpha|^2 \tag{3.32}$$

Since v is orthogonal to $\bar{m}'$ we have

$$\frac{1}{2m_\beta}|\alpha|\,\|\bar{m}'\|_2 = \frac{1}{2m_\beta}\int w\bar{m}'\,dx \leqslant \|w\|_\infty \tag{3.33}$$

Inequality $\|w\|_\infty^2 \leqslant 2\,\|w\|_2\,\|w'\|_2$, (3.32) and (3.33) imply

$$\|v'\|_2 \leqslant c(\beta, J)\,\|w\|_2^{1/2}\,\|w'\|_2^{1/2} \tag{3.34}$$

where $c(\beta, J)$ is a positive constant. Then applying (3.18), since $\|Pv'\|_2 = \|w'\|_2$ and (3.34) we have

$$\varepsilon\gamma\sigma(m_\beta)\,\|w'\|^2 - \frac{1}{\varepsilon}\int \sigma(\bar{m})[U(v)]^2\,dx$$

$$\geqslant \varepsilon\gamma\sigma(m_\beta)\,\|w'\|_2^2 - \frac{c(\varepsilon_0, \beta, J)}{\varepsilon}\,\|v'\|_2^4 - \frac{\varepsilon_0}{\varepsilon}\,\|w'\|_2^2$$

$$\geqslant \left[\varepsilon\gamma\sigma(m_\beta) - \frac{\varepsilon_0}{\varepsilon}\right]\|w'\|_2^2 - \frac{c(\varepsilon_0, \beta, J)}{\varepsilon}\,\|w\|_2^2\,\|w'\|_2^2$$

$$= \|w'\|_2^2\left[\varepsilon\gamma\sigma(m_\beta) - \frac{\varepsilon_0}{\varepsilon} - \frac{2c(\varepsilon_0, \beta, J)}{\varepsilon}\,\|w\|_2^2\right] \tag{3.35}$$

Since (3.25) the last term of (3.35) is bigger or equal to

$$\varepsilon\gamma\sigma(m_\beta) - \frac{\varepsilon_0}{\varepsilon} - \frac{2c(\varepsilon_0, \beta, J)}{\varepsilon}\left[\|v\|_2^2 + \frac{\|v'\|_2^2}{\|\bar{m}''\|_2^2}\right] \tag{3.36}$$

Take $\varepsilon_0 = \varepsilon^{2+r}$. We can always find $r > 0$, $\delta_2(\varepsilon, \beta, J) > 0$ and $\kappa(\varepsilon, \beta, J) > 0$ such that provide $\|v\|_2 \leqslant \delta_2(\varepsilon, \beta, J)$ and $\|v'\|_2 \leqslant \kappa(\varepsilon, \beta, J)$ the term in (3.36) is positive. This is the only place where the bound on $\|v'\|_2$ depends on the chosen ε. ▮

Remark. In the case v is antisymmetric, since v' is symmetric and $\bar{m}''$ is antisymmetric the projection operator P in Lemma 3.4 is equal to I, the identity operator. Then, in this case, Theorem 3.2 follows applying simply (3.17) of Lemma 3.5.

Proof of Theorem 3.1. The proof of theorem follows applying Theorem 3.2 and Lemma 1.2.

Since $\mathscr{F}$ is decreasing along the solution of (1.12), see Theorem 2.1, we have

$$\frac{d\mathscr{F}}{dt}(m(t)) = -\mathscr{I}(m(t)) \tag{3.37}$$

We take k and δ_0 in the statement of the theorem to be respectively $k \leqslant k(\beta, J, \frac{1}{6})$ where $k(J, \beta, \frac{1}{6})$ is the quantity in Theorem 3.2 for $\varepsilon = \frac{1}{6}$ and $\delta_0 \leqslant \min\{\delta(J, \beta, \frac{1}{6});\ \ \delta(k, \beta, J)\}$ where $\delta(J, \beta, \frac{1}{6})$ is the quantity in Theorem 3.2 for $\varepsilon = \frac{1}{6}$ and $\delta(k, \beta, J)$ is the quantity in the statement of Lemma 1.2.

We apply Theorem 3.2, for $\varepsilon = \frac{1}{6}$, and the uncertainty principle (1.36) to (3.37) obtaining

$$\frac{d\mathcal{F}}{dt}(m(t)) \leqslant -\frac{1}{2}\sigma(m_\beta)\frac{\|\mathscr{A}(m(t)-\bar{m})\|_2^4}{\|x\mathscr{A}(m(t)-\bar{m})\|_2^2}$$

Since $\mathscr{A}$ has a gap, see (1.24)

$$\frac{d\mathcal{F}}{dt}(m(t)) \leqslant -\frac{1}{2}\sigma(m_\beta)\,\alpha^2\frac{\|(m(t)-\bar{m})\|_2^4}{\|x\mathscr{A}(m(t)-\bar{m})\|_2^2} \tag{3.38}$$

Applying Lemma 1.2 to (3.38) and recalling the definition of ϕ, (1.39), we have

$$\frac{d\mathcal{F}}{dt}(m(t)) \leqslant -\frac{1}{2}\sigma(m_\beta)^2\frac{\alpha^2}{C}\frac{[\mathcal{F}(m(t))-\mathcal{F}(\bar{m})]^2}{\phi(t)} \tag{3.39}$$

Denote by $A = \frac{1}{2}\sigma(m_\beta)^2\,\alpha^2/C$ which obviously depends on κ, δ_0, J, β and the theorem is proved. ∎

4. MOMENT ESTIMATES

Our goal in this section is to estimate the growth of

$$\phi(t) = \int \sigma(\bar{m})\,|x\mathscr{A}v|^2\,dx \tag{4.1}$$

where v is a solution of (1.34), and the mobility $\sigma(\bar{m})$ is given by (1.6) evaluated at $\bar{m}$. Recall that our notational convention is that $\bar{m}$ stands for $\bar{m}_{a(t)}$ and that $\mathscr{A}$ denotes the second variation of $\mathcal{F}$ at $\bar{m}_{a(t)}$. Thus, when we differentiate (4.1), we must take this into account. Moreover we will need pointwise bounds on functions such as $x^2\bar{m}'(x)$ in what follows. Clearly $\|x^2\bar{m}'(x)\|_\infty \leqslant \infty$ by (1.10) but the bound depends on $|a(t)|$, since $(2m_\beta)^{-1}\,\bar{m}'(x)$ is a probability density centered on $a(t)$. For this reason, our hypotheses involve a bound on $|a(t)|$.

Theorem 4.1. Let v be a solution of (1.34). Then for any κ, there are finite constants $\delta = \delta(\kappa, \beta, J) > 0$ and $C = C(\kappa, \beta, J) < \infty$ such that for all t with $\|v'(t)\|_2 \leqslant \kappa$, $\|v(t)\|_2 \leqslant \delta$, and $|a(t)| \leqslant 1$.

$$\frac{d}{dt}\phi(t) \leqslant C[\mathcal{F}(\bar{m}+v) - \mathcal{F}(\bar{m})] \tag{4.2}$$

Proof. Since $\mathscr{A}$ is self adjoint,

$$\frac{d}{dt}\int \sigma(\bar{m})\,|x\mathscr{A}v|^2\,dx$$

$$= 2\int \mathscr{A}(\sigma(\bar{m})\,x^2\mathscr{A}v)\,\frac{\partial v}{\partial t}\,dx$$

$$+\,\dot{a}(t)\,2\left(\beta\int \bar{m}\bar{m}'x^2\,|\mathscr{A}v|^2\,dx+\int \sigma(\bar{m})\,x^2(\mathscr{A}v)\,\frac{2\bar{m}\bar{m}'}{\beta(1-\bar{m}^2)^2}\,v\,dx\right) \qquad (4.3)$$

We arrange the proof in a series of lemmas. The terms in (4.3) involving $\dot{a}(t)$ are the easiest to estimate, and we begin with these.

Lemma 4.2. Let v be a solution of (1.34). Then for any $\varepsilon > 0$, there is a constant $\delta = \delta(\kappa, \beta, J) > 0$ such that for all t with $\|v(t)\|_2 \leqslant \delta(\kappa, \beta, J)$

$$\frac{d}{dt}\,\phi(t) \leqslant 2\int \mathscr{A}(\sigma(\bar{m})\,x^2\mathscr{A}v)\,\frac{\partial v}{\partial t}\,dx + \varepsilon[\mathscr{F}(\bar{m}+v)-\mathscr{F}(\bar{m})] \qquad (4.4)$$

Proof. By (1.10) and the boundedness of $\mathscr{A}$ on L^2 and then Theorem 2.4, which says that $|\dot{a}(t)| \leqslant C\,\|v(t)\|_2$, one clearly has that

$$\dot{a}(t)\,2\left(\beta\int \bar{m}\bar{m}'x^2\,|\mathscr{A}v|^2\,dx+\int \sigma(\bar{m})\,x^2(\mathscr{A}v)\,\frac{2\bar{m}\bar{m}'}{\beta(1-\bar{m}^2)^2}\,v\,dx\right)$$

$$\leqslant |\dot{a}(t)|\;C\,\|v(t)\|_2^2 \leqslant C\,\|v(t)\|_2^3 \qquad (4.5)$$

where C is a constant depending only on β and J that changes from line to line. The result clearly follows from this and Theorem 2.4.

Now for the part of (4.3) coming from the time derivative of v, recall that

$$\frac{\partial v}{\partial t} = (\sigma(\bar{m})(\mathscr{A}v)')' + \beta(v^2 J * \bar{m}')' + \beta(v(v+2\bar{m})\,J * v')' + \dot{a}\bar{m}' \qquad (4.6)$$

and we will separately estimate the linear and nonlinear contributions from (4.6) to (4.3). Note that since $\mathscr{A}\bar{m}' = 0$, the term containing $\dot{a}$ in (4.6) makes no contribution to (4.3).

The basic manipulation, to be done repeatedly in the rest of the proof, is to commute differentiation and multiplication by x with $\mathscr{A}$. Therefore we define

$$g(x) = \frac{2\bar{m}\bar{m}'}{(1-\bar{m}^2)^2} \qquad (4.7)$$

and observe that for any function w,

$$(\mathscr{A}w)' = \mathscr{A}w' + gw \tag{4.8}$$

Furthermore, define the convolution operator $\mathscr{C}$ by

$$\mathscr{C}w(x) = \int J(y)\, yw(x-y)\, dy \tag{4.9}$$

and observe that for any function w,

$$x(\mathscr{A}w) = \mathscr{A}(xw) + \mathscr{C}w \tag{4.10}$$

where xw denotes the function with values $xw(x)$. Note that by Young's inequality $\mathscr{C}$ is bounded on all L^p with operator norm

$$\|\mathscr{C}\| \leqslant \int |xJ|\, dx \tag{4.11}$$

We shall also need the following technical lemma:

Lemma 4.3. For any function w,

$$\|\sigma(\bar{m})\, x(\mathscr{A}w)'\|_2 \leqslant \frac{\|\sigma(\bar{m})\, x\bar{m}'\|_2}{\|\bar{m}'\|_2} \|w\|_2 + \alpha^{-1/2} \|\mathscr{A}^{1/2}(\sigma(\bar{m})\, x(\mathscr{A}w)')\|_2 \tag{4.12}$$

where α is the spectral gap (1.24) of $\mathscr{A}$. Also, there is a finite constant $K(\beta, J)$ depending only on β and J such that whenever $|a(t)| \leqslant 1$,

$$\|J * (xw')\|_2 \leqslant K(\|w\|_2 + \|\mathscr{A}^{1/2}(\sigma(\bar{m})\, x(\mathscr{A}w)')\|_2) \tag{4.13}$$

Proof. Let P denote the orthogonal projection onto the span of $\bar{m}'$; i.e., the null space of $\mathscr{A}$. Then

$$\begin{aligned} P(\sigma(\bar{m})\, x(\mathscr{A}w)') &= \frac{1}{\|\bar{m}'\|_2^2} \langle \bar{m}', \sigma(\bar{m})\, x(\mathscr{A}w)' \rangle_{L^2} \bar{m}' \\ &= -\frac{1}{\|\bar{m}'\|_2^2} \langle (\sigma(\bar{m})\, x\bar{m}')', (\mathscr{A}w) \rangle_{L^2} \bar{m}' \end{aligned}$$

Hence, by the Schwarz inequality,

$$\|P(\sigma(\bar{m})\, x(\mathscr{A}w)')\|_2 \leqslant \frac{\|(\sigma(\bar{m})\, x\bar{m}')'\|_2}{\|\bar{m}'\|_2} \|\mathscr{A}w\|_2 \tag{4.14}$$

Next,

$$\begin{aligned}\|P^{\perp}(\sigma(\bar{m})\, x(\mathscr{A}w)')\|_2 &= \|\mathscr{A}^{-1/2}\mathscr{A}^{1/2}P^{\perp}(\sigma(\bar{m})\, x(\mathscr{A}w)')\|_2\\ &\leqslant \alpha^{-1/2}\, \|\mathscr{A}^{1/2}P^{\perp}(\sigma(\bar{m})\, x(\mathscr{A}w)')\|_2\\ &= \alpha^{-1/2}\, \|P^{\perp}\mathscr{A}^{1/2}(\sigma(\bar{m})\, x(\mathscr{A}w)')\|_2\\ &\leqslant \alpha^{-1/2}\, \|\mathscr{A}^{1/2}(\sigma(\bar{m})\, x(\mathscr{A}w)')\|_2\end{aligned}$$

Hence, the Minkowski inequality and (4.14) yield the result.

The proof of (4.13) is more involved. To begin, define the operator $\mathscr{B}$ by

$$\mathscr{B}w = \frac{1}{\beta(1-m_\beta^2)}\, w - J * w$$

Fourier transforming, one sees that $\mathscr{B}$ is bounded with a bounded inverse since $\beta(1-m_\beta^2) < 1$. Also, $\mathscr{B}$ commutes with convolution by J, and

$$x\mathscr{B}w = \mathscr{B}(xw) + \mathscr{C}w$$

as with $\mathscr{A}$ in (4.10). Hence,

$$\begin{aligned}J * xv' &= \mathscr{B}^{-1}J * (\mathscr{B}xv') = \mathscr{B}^{-1}J * (x\mathscr{B}v' - \mathscr{C}v')\\ &= -\mathscr{B}^{-1}\mathscr{C}(J' * v) + \mathscr{B}^{-1}J * (x\mathscr{A}v' - \tilde{g}v')\end{aligned}$$

where

$$\tilde{g} = \frac{1}{\beta(1-\bar{m}^2)} - \frac{1}{\beta(1-m_\beta^2)}$$

and where we have used the fact that convolution with J commutes with $\mathscr{C}$ and that $J * v' = J' * v$. Next, using (4.8),

$$\mathscr{B}^{-1}J * (x\mathscr{A}v' - \tilde{g}v') = \mathscr{B}^{-1}J * (x(\mathscr{A}v)') + (xg - \tilde{g})\, v')$$

and

$$\begin{aligned}\mathscr{B}^{-1}J * (((xg - \tilde{g})\, v') &= \mathscr{B}^{-1}J * (((xg - \tilde{g})\, v)') - \mathscr{B}^{-1}J * ((xg - \tilde{g})'\, v)\\ &= \mathscr{B}^{-1}J' * (((xg - \tilde{g})\, v)) - \mathscr{B}^{-1}J * ((xg - \tilde{g})'\, v)\end{aligned}$$

where again we have once again used $J * v' = J' * v$. Thus,

$$\|J * (xv')\|_2 \leqslant \|\mathscr{B}^{-1}\mathscr{C}(J' * v)\|_2 + \|\mathscr{B}^{-1}J * (x(\mathscr{A}v)')\|_2$$
$$+ \|\mathscr{B}^{-1}J' * (((xg - \tilde{g})\, v))\|_2 + \|\mathscr{B}^{-1}J * ((xg - \tilde{g})'\, v)\|_2$$
$$\leqslant C(\beta, J)(\|v\|_2 + \|\sigma(\bar{m})\, x(\mathscr{A}v)'\|_2$$

since $(xg - \tilde{g})'$ and $(xg - \tilde{g})'$ are bounded by (1.10) and the hypothesis that $|a(t)| \leqslant 1$. Now application of (4.12) yields (4.13). ∎

We next estimate the nonlinear contribution from (4.6) to (4.3).

Lemma 4.4. Let v be a solution of (1.34). Then for any $\varepsilon > 0$, there are constants $\delta = \delta(\beta, J, \varepsilon) > 0$ and $\kappa = \kappa(\beta, J, \varepsilon) > 0$ such that for all t with $\|v(t)\|_2 \leqslant \delta$, $\|v'(t)\|_2 \leqslant \kappa$, and $|a(t)| \leqslant 1$,

$$\frac{d}{dt}\phi(t) \leqslant 2 \int \mathscr{A}(\sigma(\bar{m})\, x^2 \mathscr{A}v)(\sigma(\bar{m})(\mathscr{A}v)')'\, dx$$
$$+ \varepsilon[\mathscr{F}(\bar{m} + v) - \mathscr{F}(\bar{m})] + \varepsilon\, \|\mathscr{A}^{1/2}(\sigma(\bar{m})\, x(\mathscr{A}v)')\|_2^2 \tag{4.15}$$

Proof. We separately estimate the contribution of the two nonlinear terms in (4.6) to (4.3), beginning with the more difficult of the two:

$$2 \int \mathscr{A}(\sigma(\bar{m})\, x^2 \mathscr{A}v)\, \beta(v(v + 2\bar{m})\, J * v')'\, dx \tag{4.16}$$

Now integrating by parts and applying (4.8) to (4.16) yields

$$-2 \int g(\sigma(\bar{m})\, x^2 \mathscr{A}v)\, \beta(v(v + 2\bar{m})\, J' * v)\, dx$$
$$-2 \int \mathscr{A}(\sigma(\bar{m})'\, x^2 \mathscr{A}v)\, \beta(v(v + 2\bar{m})\, J' * v)\, dx$$
$$-4 \int \mathscr{A}(\sigma(\bar{m})\, x \mathscr{A}v)\, \beta(v(v + 2\bar{m})\, J * v')\, dx$$
$$-2 \int \mathscr{A}(\sigma(\bar{m})\, x^2 (\mathscr{A}v)')\, \beta(v(v + 2\bar{m})\, J * v')\, dx \tag{4.17}$$

where the first term on the right comes from (4.8), and the other three from differentiating the product $\sigma(\bar{m})\, x^2 \mathscr{A}v$. We have also used the fact that

$J * v' = J' * v$. Because of (1.10), the two integrals in (4.17) that are written in terms of $J' * v$ can easily be estimated above by

$$C \|v\|_\infty \|v\|_2^2 \tag{4.18}$$

where C is a constant depending only on β and J. Then by Lemma A.2 and $\|v\|_\infty^2 \leqslant 2 \|v'\|_2 \|v\|_2$, there are constants $\delta > 0$ and $\kappa > 0$ such that for all t with $\|v(t)\|_2 \leqslant \delta$, $\|v'(t)\|_2 \leqslant \kappa$ and $|a(t)| \leqslant 1$, the quantity in (4.18) is no greater than

$$\frac{\varepsilon}{3} [\mathscr{F}(\bar{m}+v) - \mathscr{F}(\bar{m})] \tag{4.19}$$

For the remaining integrals in (4.17), we need to commute an x past $\mathscr{A}$. Applying (4.10), these become

$$-4 \int \mathscr{C}(\sigma(\bar{m}) \mathscr{A} v) \beta(v(v+2\bar{m}) J * v') \, dx$$

$$-2 \int \mathscr{C}(\sigma(\bar{m}) x(\mathscr{A} v)') \beta(v(v+2\bar{m}) J * v') \, dx$$

$$-4 \int \mathscr{A}(\sigma(\bar{m}) \mathscr{A} v) \beta(v(v+2\bar{m}) xJ * v') \, dx$$

$$-2 \int \mathscr{A}(\sigma(\bar{m}) x(\mathscr{A} v)') \beta(v(v+2\bar{m}) xJ * v') \, dx$$

Now, it is exactly the convolution by J in $\mathscr{A}$ that doesn't commute with multiplication by x so that

$$xJ * w = J * (xw) + \mathscr{C} w$$

so that the integrals above can be partially rewritten as

$$-4 \int \mathscr{C}(\sigma(\bar{m}) \mathscr{A} v) \beta(v(v+2\bar{m}) J' * v) \, dx$$

$$-2 \int \mathscr{C}(\sigma(\bar{m}) x(\mathscr{A} v)') \beta(v(v+2\bar{m}) J' * v) \, dx$$

$$-4 \int \mathscr{A}(\sigma(\bar{m}) \mathscr{A} v) \beta(v(v+2\bar{m}) \mathscr{C} v') \, dx$$

$$-2 \int \mathscr{A}(\sigma(\bar{m}) x(\mathscr{A} v)') \beta(v(v+2\bar{m}) \mathscr{C} v') \, dx$$

$$-4\int \mathscr{A}(\sigma(\bar{m})\,\mathscr{A}v)\,\beta(v(v+2\bar{m})\,J*(xv'))\,dx$$

$$-2\int \mathscr{A}(\sigma(\bar{m})\,x(\mathscr{A}v)')\,\beta(v(v+2\bar{m})\,J*(xv'))\,dx$$

Clearly there is a constant C depending only on β and J so that

$$\|\mathscr{C}v'\|_2 \leqslant C\,\|v\|_2$$

and hence the four terms containing $\mathscr{C}$ may be estimated by

$$C\,\|v\|_\infty\,\|v\|_2^2$$

and hence by

$$\frac{\varepsilon}{3}\,[\mathscr{F}(\bar{m}+v)-\mathscr{F}(\bar{m})] \tag{4.20}$$

for all t with $\|v(t)\|_2 \leqslant \delta$, $\|v'\|_2 \leqslant \kappa$ and $|a(t)| \leqslant 1$.

Next, by the Schwarz inequality, and then (4.13) of Lemma 4.3,

$$\begin{aligned}
&-4\int \mathscr{A}(\sigma(\bar{m})\,\mathscr{A}v)\,\beta(v(v+2\bar{m})\,J*(xv'))\,dx\\
&\quad\leqslant C\,\|v\|_\infty\,\|v\|_2\,\|J*(xv')\|_2\\
&\quad\leqslant C\,\|v\|_\infty\,\|v\|_2\,(\|v\|_2+\|\mathscr{A}^{1/2}\sigma(\bar{m})\,x(\mathscr{A}v)'\|_2)
\end{aligned} \tag{4.21}$$

and

$$\begin{aligned}
&-2\int \mathscr{A}(\sigma(\bar{m})\,x(\mathscr{A}v)')\,\beta(v(v+2\bar{m})\,J*(xv'))\,dx\\
&\quad\leqslant C\,\|v\|_\infty\,\|\mathscr{A}^{1/2}\sigma(\bar{m})\,x(\mathscr{A}v)'\|_2\,\|J*(xv')\|_2\\
&\quad\leqslant C\,\|v\|_\infty\,\|\mathscr{A}^{1/2}\sigma(\bar{m})\,x(\mathscr{A}v)'\|_2\,(\|v\|_2+\|\mathscr{A}^{1/2}\sigma(\bar{m})\,x(\mathscr{A}v)'\|_2)
\end{aligned} \tag{4.22}$$

where C is a constant depending only on β and J.

Hence the sum of the two terms in (4.21) and (4.22) is no greater than

$$C\,\|v\|_\infty\,(\|v\|_2^2+\|\sigma(\bar{m})\,x(\mathscr{A}v)'\|_2^2)$$

and now decreasing δ and κ as necessary, we obtain as before from $\|v\|_\infty^2 \leqslant 2\,\|v'\|_2\,\|v\|_2$ and Lemma A.2 that this is no grater than

$$\frac{\varepsilon}{3}\,[\mathscr{F}(\bar{m}+v)-\mathscr{F}(\bar{m})]+\varepsilon\,\|\mathscr{A}^{1/2}\sigma(\bar{m})\,x(\mathscr{A}v)'\|_2^2 \tag{4.23}$$

for all t with $\|v(t)\|_2 \leqslant \delta$, $\|v'(t)\|_2 \leqslant \kappa$ and $|a(t)| \leqslant 1$. Thus the estimate on (4.16) follows from (4.19) (4.20) and (4.23).

It remains to estimate the contributions to (4.3) from the other of the two non-linear terms in (4.6), namely

$$-2 \int (\mathscr{A}(\sigma(\bar{m})\, x^2 \mathscr{A} v))' \, (v^2 J * \bar{m}')\, dx \tag{4.24}$$

Proceeding as above, though with much less effort, one obtains that there is a constant $C(\beta, J)$ depending only on β and J so that this is bounded by

$$\|v\|_\infty \, (C(\beta, J)\, \|v\|_2^2 + \|\mathscr{A}^{1/2} \sigma(\bar{m})\, x(\mathscr{A} v)'\|_2^2) \tag{4.25}$$

where the extra factor of $\|v\|_\infty$ comes from the nonlinearity. Using once more the inequality $\|v\|_\infty^2 \leqslant 2\, \|v\|_2\, \|v'\|_2$, one sees that for δ sufficiently small, one can combine the above estimates, once more using Lemma 1.2, to obtain the proof of Lemma 4.4.

Theorem 4.5. Let v be a solution of (1.34). Then for any $\varepsilon > 0$, there are constants $\delta = \delta(\beta, J, \varepsilon) > 0$ and $\kappa = \kappa(\beta, J, \varepsilon) > 0$ such that for all t with $\|v(t)\|_2 \leqslant \delta$, $\|v'(t)\|_2 \leqslant \kappa$ and $|a(t)| \leqslant 1$,

$$\begin{aligned} \frac{d}{dt}\, \phi(t) \leqslant & -4 \int \mathscr{A}(\sigma(\bar{m})\, \mathscr{A} v)(\sigma(\bar{m})\, x(\mathscr{A} v)')\, dx \\ & -2\, \|\mathscr{A}^{1/2}(\sigma(\bar{m})\, x(\mathscr{A} v)'\|_2^2 + I_1 + I_2 + I_3 + I_4 \\ & + \varepsilon[\mathscr{F}(\bar{m} + v) - \mathscr{F}(\bar{m})] + \varepsilon\, \|\mathscr{A}^{1/2}(\sigma(\bar{m})\, x(\mathscr{A} v)'\|_2^2 \end{aligned} \tag{4.26}$$

where

$$I_1 = -2 \int g(\sigma(\bar{m})\, x^2 \mathscr{A} v)(\sigma(\bar{m})(\mathscr{A} v)')\, dx$$

$$I_2 = -2 \int \mathscr{A}(\sigma(\bar{m})'\, x^2 \mathscr{A} v)(\sigma(\bar{m})(\mathscr{A} v)')\, dx$$

$$I_3 = -4 \int \mathscr{C}(\sigma(\bar{m})\, \mathscr{A} v)(\sigma(\bar{m})(\mathscr{A} v)')\, dx$$

$$I_4 = -2 \int \mathscr{C}(\sigma(\bar{m})\, x(\mathscr{A} v)')(\sigma(\bar{m})(\mathscr{A} v)')\, dx$$

Proof. First, define L by

$$L = 2\int \mathscr{A}(\sigma(\bar{m})\, x^2 \mathscr{A} v)(\sigma(\bar{m})(\mathscr{A} v)')'\, dx$$

$$= -2\int (\mathscr{A}(\sigma(\bar{m})\, x^2 \mathscr{A} v))'\, (\sigma(\bar{m})(\mathscr{A} v)')\, dx \tag{4.27}$$

Now applying (4.8) to (4.27) yields

$$\begin{aligned} L = &-2\int g(\sigma(\bar{m})\, x^2 \mathscr{A} v)(\sigma(\bar{m})(\mathscr{A} v)')\, dx \\ &-2\int \mathscr{A}(\sigma(\bar{m})'\, x^2 \mathscr{A} v)(\sigma(\bar{m})(\mathscr{A} v)')\, dx \\ &-4\int \mathscr{A}(\sigma(\bar{m})\, x \mathscr{A} v)(\sigma(\bar{m})(\mathscr{A} v)')\, dx \\ &-2\int \mathscr{A}(\sigma(\bar{m})\, x^2 (\mathscr{A} v)')(\sigma(\bar{m})(\mathscr{A} v)')\, dx \end{aligned} \tag{4.28}$$

where the first term on the right comes from (4.8), and the other three from differentiating the product $\sigma(\bar{m})\, x^2 \mathscr{A} v$.

Denote the first two terms in the expression obtained for L in (4.28) by I_1 and I_2 respectively, so that

$$\begin{aligned} L = I_1 + I_2 &- 4\int \mathscr{A}(\sigma(\bar{m})\, x \mathscr{A} v)(\sigma(\bar{m})(\mathscr{A} v)')\, dx \\ &-2\int \mathscr{A}(\sigma(\bar{m})\, x^2 (\mathscr{A} v)')(\sigma(\bar{m})(\mathscr{A} v)')\, dx \end{aligned}$$

Next, to exploit the positivity of $\mathscr{A}$, we need to distribute the factors of x symmetrically in the last integral. To do this, apply (4.10) to account for commuting multiplication by x with $\mathscr{A}$. We also do this in the other integral, so that the same function $\sigma(\bar{m})\, x(\mathscr{A} v)'$ is produced there as well. The result is

$$\begin{aligned} L = I_1 + I_2 &- 4\int \mathscr{C}(\sigma(\bar{m})\, \mathscr{A} v)(\sigma(\bar{m})(\mathscr{A} v)')\, dx \\ &-2\int \mathscr{C}(\sigma(\bar{m})\, x (\mathscr{A} v)')(\sigma(\bar{m})(\mathscr{A} v)')\, dx \end{aligned}$$

$$-4\int \mathscr{A}(\sigma(\bar{m})\,\mathscr{A}v)(\sigma(\bar{m})\,x(\mathscr{A}v)')\,dx$$

$$-2\int \mathscr{A}(\sigma(\bar{m})\,x(\mathscr{A}v)')(\sigma(\bar{m})\,x(\mathscr{A}v)')\,dx$$

Now denote the first two terms after I_1 and I_2; i.e., those containing $\mathscr{C}$, by I_3 and I_4 respectively. Then, by Lemma 4.4, the result is proved. ▮

Proof of Theorem 4.1. The first two terms in (4.26) are the key to the analysis. They correspond to the two terms produced in (1.37) when similar estimates were performed on the heat equation as an illustration of the method. To see this more easily, introduce the following notations:

$$f=\sigma(\bar{m})\,\mathscr{A}v \tag{4.29}$$

and

$$h=\sigma(\bar{m})\,x(\mathscr{A}v)' \tag{4.30}$$

Then these integrals can be written and estimated as

$$\begin{aligned}
&-4\langle f, \mathscr{A}h\rangle-2\langle h, \mathscr{A}h\rangle\\
&\quad=-\langle h+2f, \mathscr{A}(h+2f)\rangle-\langle h, \mathscr{A}h\rangle+4\langle f, \mathscr{A}f\rangle\\
&\quad\leqslant-\langle h, \mathscr{A}h\rangle+4\langle f, \mathscr{A}f\rangle
\end{aligned}$$

Thus,

$$L\leqslant\sum_{j=1}^{4} I_j-\langle h, \mathscr{A}h\rangle+4\langle f, \mathscr{A}f\rangle \tag{4.31}$$

The next step is to-estimate each of the I_j in terms of $\|v\|_2^2$, using the negative term in (4.31) to absorb contributions from v'.

First, using the Schwarz inequality, and then the arithmetic-geometric mean inequality,

$$\begin{aligned}
I_1&\leqslant 2\,\|g\sigma(\bar{m})\,x^2\mathscr{A}v\|_2\,\|\sigma(\bar{m})(\mathscr{A}v)'\|_2\\
&\quad 2\lambda\,\|g\sigma(\bar{m})\,x^2\mathscr{A}v\|_2+\frac{1}{\lambda}\,\|\sigma(\bar{m})(\mathscr{A}v)'\|_2
\end{aligned}$$

for any $\lambda>0$. Now choose λ so large that the estimate (4.12) of Lemma 4.3 gives

$$\frac{1}{\lambda}\,\|\sigma(\bar{m})(\mathscr{A}v)'\|_2\leqslant\frac{1}{4}\,(\|v\|_2^2+\langle h, \mathscr{A}h\rangle_{L^2})$$

where h is given in (4.30). One obtains a constant C depending only on β, and J such that

$$I_1 \leqslant \tfrac{1}{4}\langle h, \mathscr{A}h\rangle + C\,\|v\|_2^2 \tag{4.32}$$

It is easier to deal with I_2. Schwarz and (1.10) suffice to establish that there is a constant C depending only on β, and certain finite moments of $\bar{m}'$ so that

$$I_2 \leqslant \tfrac{1}{4}\langle h, \mathscr{A}h\rangle + C\,\|v\|_2^2 \tag{4.33}$$

To bound I_3, we will integrate by parts. Note that using (4.29)

$$\begin{aligned} I_3 &= -4\int (\mathscr{C}f)\,\sigma(\bar{m})(\mathscr{A}v)'\,dx \\ &= 4\int (\mathscr{C}f)'\,\sigma(\bar{m})(\mathscr{A}v)\,dx + 4\int (\mathscr{C}f)\,\sigma(\bar{m})'\,(\mathscr{A}v)\,dx \end{aligned}$$

Note that by Young's inequality the operator $(d/dx)\circ\mathscr{C}$ is bounded by $\int |xJ'(x) + J(x)|\,dx$ on all L^p, in particular L^2. Using this, (4.11) and the rapid decay of $\sigma(\bar{m})'$ coming from (1.10), there is clearly a constant C depending only on β and J so that

$$I_3 \leqslant C\,\|\sigma(\bar{m})\,\mathscr{A}v\|_2^2 \tag{4.34}$$

Finally, to bound I_4, we use (4.30) and again integrate by parts:

$$\begin{aligned} I_4 &= -2\int (\mathscr{C}h)\,\sigma(\bar{m})(\mathscr{A}v)'\,dx \\ &= 2\int (\mathscr{C}h)'\,\sigma(\bar{m})(\mathscr{A}v)\,dx + 2\int (\mathscr{C}h)\,\sigma(\bar{m})'\,(\mathscr{A}v)\,dx \end{aligned}$$

Now proceeding as with I_3, one obtains a constant C depending only on β and J so that

$$I_4 \leqslant C\,\|\sigma(\bar{m})\,\mathscr{A}v\|_2\,\langle h, \mathscr{A}h\rangle^{1/2} \leqslant \tfrac{1}{4}\langle h, \mathscr{A}h\rangle + 4C^2\,\|\sigma(\bar{m})\,\mathscr{A}v\|_2^2 \tag{4.35}$$

Then, inserting (4.32), (4.33), (4.34) and (4.35) into (4.31), and using the fact that $\mathscr{A}$ is bounded, with a bound depending only on β and J, one obtains

$$L \leqslant C\,\|v\|_2^2 - \tfrac{1}{4}\langle h, \mathscr{A}h\rangle \tag{4.36}$$

where C is a constant depending only on β and J. Then for $\varepsilon < 1/4$, Lemma 4.4 and again Lemma 1.2, we have the result.

5. SYSTEM OF DIFFERENTIAL INEQUALITIES AND RELAXATION BOUNDS

We begin this section by proving the bounds stated in the introduction for solutions of (1.38).

Theorem 5.1. Let f and ϕ be any two non-negative solutions of the system of differential inequalities

$$\begin{aligned} \frac{d}{dt} f(t) &\leqslant -A\,\frac{f(t)^2}{\phi(t)} \\ \frac{d}{dt} \phi(t) &\leqslant Bf(t) \end{aligned} \tag{5.1}$$

Then

$$\begin{aligned} f(t) &\leqslant f(0)^{1-q} \left(\frac{\phi(0)}{A+B}\right)^q \left(\frac{\phi(0)}{(A+B)\, f(0)} + t\right)^{-q} \\ \phi(t) &\leqslant (A+B)\, f(0))^{1-q}\, \phi(0)^q \left(\frac{\phi(0)}{(A+B)\, f(0)} + t\right)^{1-q} \end{aligned} \tag{5.2}$$

where

$$q = \frac{A}{A+B} \tag{5.3}$$

Proof. One easily checks that for any positive constants a and b, the functions

$$F(t) = b(a+t)^{-q} \qquad \text{and} \qquad \Phi(t) = b(A+B)(a+t)^{1-q}$$

satisfy the system (5.1) with equality holding instead of inequality for any values of C and a, as long a q is given by (5.3). Choose C and a so that $f(0) < F(0)$ and $\phi(0) < \Phi(0)$. The first inequality (5.1) can be written as

$$\frac{d}{dt}\frac{1}{f(t)} \geqslant A\,\frac{1}{\phi(t)}$$

so that

$$\frac{1}{f(t)} \geqslant \frac{1}{f(0)} + A \int_0^t \frac{1}{\phi(s)}\, ds$$

In this way one obtains

$$f(t) \leqslant \left(A \int_0^t \frac{1}{\phi(s)}\, ds + \frac{1}{f(0)} \right)^{-1}$$

$$\phi(t) \leqslant B \int_0^t f(s)\, ds + \phi(0)$$

Now suppose that there exists some t such that either $f(t) \geqslant F(t)$ or $\phi(t) \geqslant \Phi(t)$. Then, there would be a first such time, and by our assumptions on the initial conditions this first time must be strictly positive. Let t denote this putative first time, so that

$$f(s) \leqslant F(s) \qquad \text{and} \qquad \phi(s) \leqslant \Phi(s) \tag{5.4}$$

for all $s \leqslant t$ Then

$$f(t) \leqslant \left(A \int_0^t \frac{1}{\phi(s)}\, ds + \frac{1}{f(0)} \right)^{-1} < \left(A \int_0^t \frac{1}{\Phi(s)}\, ds + \frac{1}{F(0)} \right)^{-1} = F(t)$$

where the inequality is strict since $f(0) < F(0)$. In the same way one deduces that $\phi(t) < \Phi(t)$. This contradiction establishes that (5.4) holds for all $s \geqslant 0$. Moreover, since the amounts by which $f(0) < F(0)$ and $\phi(0) < \Phi(0)$ were arbitrary, (5.4) still holds for all $s \geqslant 0$ in the limiting case in which $f(0) = F(0)$ and $\phi(0) = \Phi(0)$. These two conditions fix the values of a and b, and the theorem is proved. ∎

Proof of Theorem 1.3. First, choose $\delta_1 > 0$ and $\kappa_1 > 0$ such that the estimates of Lemma 1.2, Theorem 3.1, and Theorem 4.1 all hold with finite constants A, B and C. Further decrease κ_1 and δ_1, if need be, so that by Theorem 2.4, $a(t)$ is well defined and differentiable.

Next define δ_0 by

$$\delta_0 = \frac{\delta}{4(C+1)} \tag{5.5}$$

where C is the constant in Lemma 1.2. Theorem 2.2 provides us with an $\varepsilon_0 > 0$ and a $t_0 < \infty$ such that when $\|m_0 - \bar{m}\|_2 \leqslant \varepsilon_0$, the solution to (1.12) satisfies

$$\|v(t_0)\|_2 \leqslant \delta_0$$

and

$$\|v'(t)\|_2 \leqslant \kappa_1$$

for all $t \geqslant t_0$ such that $\|v(t)\|_2 \leqslant \delta$. Now define

$$T_0 = \min\{\inf\{t > t_0 \mid \|v(t_0)\|_2 \geqslant \delta_0/2\}, \inf\{t > t_0 \, |a(t)| \geqslant 1\}\}$$

Then uniformly on the interval (t_0, T_0), (5.1) holds with f and ϕ chosen as specified in the introduction, and A and B as above. Then, by the Theorem 5.1, this yields the algebraic decay of the excess free energy on the interval (t_0, T_0), since by the hypotheses of Theorem 1.1, both $f(0)$ and $\phi(0)$ are finite. Clearly, since f is monotone decreasing, $f(t_0)$ is also finite. Using Theorem 2.4, it is not hard to see that $\phi(t_0)$ is finite as well.

It remains to show that by further decreasing ε_0 if necessary, one has $T_0 = \infty$. Suppose that this is not the case. Then $\|v(T_0)\|_2 = \delta_1/2$. Since Lemma 1.2 is still valid with the same constant C on the closed interval $[t_0, T_0]$, and since the excess free energy is monotone decreasing,

$$\frac{\delta_1^2}{4} = \|v(T_0)\|_2^2 \leqslant C(\mathscr{F}(\bar{m} + v(T_0)) - \mathscr{F}(\bar{m}))$$

$$\leqslant C(\mathscr{F}(\bar{m} + v(t_0)) - \mathscr{F}(\bar{m})) \leqslant C^2 \, \|v(t_0)\|_2^2 \leqslant C^2 \delta_0^2$$

This contradicts (5.5), and hence $T_0 < \infty$ is not possible. ∎

APPENDIX

We restate in a slightly more general form and prove the Lemma 1.2:

Lemma A. For any $\kappa > 0$, there exists $\delta(\kappa) > 0$ and $C = C(\kappa, \beta, J) > 0$ such that for any function $m = \bar{m} + v$, where $\bar{m}$ is the closest instanton in L^2 to m we have

$$\frac{1}{C} \|m - \bar{m}\|_2^2 \leqslant \mathscr{F}(m) - \mathscr{F}(\bar{m}) \leqslant C \, \|m - \bar{m}\|_2^2 \tag{A.1}$$

provided $\|v\|_2 \leqslant \delta(\kappa)$ and $\|v'\|_2 \leqslant \kappa$.

Moreover for any $\varepsilon > 0$ and κ there is a $\tilde{\delta}(\varepsilon, \kappa, \beta, J)$ so that

$$\frac{1-\varepsilon}{2} \langle v, \mathscr{A} v\rangle \leqslant \mathscr{F}(m) - \mathscr{F}(\bar{m}) \leqslant \frac{1+\varepsilon}{2} \langle v, \mathscr{A} v\rangle \tag{A.2}$$

provided $\|v\|_2 \leqslant \tilde{\delta}(\varepsilon, \kappa, \beta, J)$ and $\|v'\|_2 \leqslant \kappa$.

Proof. We can represent

$$\mathscr{F}(m) - \mathscr{F}(\bar{m}) = \int_0^1 d\tau \, \mathscr{F}'(\bar{m} + \tau v)(v) = \int_0^1 d\tau \int_0^\tau ds \, \mathscr{F}''(\bar{m} + sv)\langle v, v\rangle$$

In order to get a lower bound for the last term above we expand $\mathscr{F}''(\bar{m}+sv)\langle v, v\rangle$ around $s=0$ obtaining

$$\mathscr{F}''(\bar{m}+s(m-\bar{m}))\langle v, v\rangle = \mathscr{F}''(\bar{m})\langle v, v\rangle + \mathscr{F}''(\tilde{m})\langle v, v, v\rangle$$

where $\tilde{m}=\bar{m}+s_0 v$ for some s_0 between 0 and 1 by the mean value theorem. Therefore

$$\mathscr{F}(m)-\mathscr{F}(\bar{m}) = \tfrac{1}{2}\langle \mathscr{A}v, v\rangle + \int_0^1 d\tau \int_0^\tau ds\, \mathscr{F}'''(\tilde{m})\langle v, v, v\rangle \tag{A.3}$$

Now we need a lower bound on the term involving the third derivative of the free energy. But by direct computation,

$$|\mathscr{F}'''(\tilde{m})\langle v, v, v\rangle| = \frac{2}{\beta}\left|\int_{\mathbb{R}} \frac{\tilde{m}}{(1-\tilde{m}^2)^2}(v(x))^3\, dx\right| \leqslant c_\beta \int_{\mathbb{R}} |v(x)|^3\, dx \tag{A.4}$$

for some constant c_β depending only on β.

Note that for any x,

$$v^2(x) = 2\int_{-\infty}^{x} v(r)\, v'(r)\, dr \leqslant 2\, \|v\|_2\, \|v'\|_2$$

Therefore we can always take δ such that $\|\tilde{m}\|_\infty < 1$. Moreover, since by hypothesis $\|v'\| \leqslant \kappa$, from (A.4) we have that

$$|\mathscr{F}'''(\tilde{m})\langle v, v, v\rangle| \leqslant \sqrt{2}\, c_\beta \kappa^{1/2} (\|v\|_2)^{5/2} \tag{A.5}$$

By (1.24)

$$\langle \mathscr{A}v, v\rangle \geqslant \alpha\, \|v\|_2^2$$

Therefore choosing δ sufficiently small, since $5/2 > 2$, it is clear that

$$\mathscr{F}''(\bar{m}+s(m-\bar{m}))\langle v, v\rangle \geqslant C^{-1}\, \|v\|_2^2$$

where C is a constant depending on β, J, κ and $\delta(\kappa)$. So we established a lower bound for (A.1). The upper bound follows from the boundness of $\mathscr{A}$. In this way we proved (A.1).

In a similar way the inequality (A.2) follows. Namely, from (A.3), for any positive ε and N

$$\mathscr{F}(m)-\mathscr{F}(\bar{m}) = \tfrac{1}{2}(1-\varepsilon)\langle \mathscr{A}v, v\rangle + \left[\tfrac{1}{2}\varepsilon\langle \mathscr{A}v, v\rangle + \int_0^1 d\tau \int_0^\tau ds\, \mathscr{F}'''(\tilde{m})\langle v, v, v\rangle\right] \tag{A.6}$$

From (1.24) and (A.5) the last term in (A.6) is bigger or equal to

$$\tfrac{1}{2}\varepsilon\alpha \,\|v\|_2^2 - \sqrt{2}\, c_\beta \kappa^{1/2}(\|v\|_2)^{5/2} = \|v\|_2^2 \,[\tfrac{1}{2}\varepsilon\alpha - \sqrt{2}\, c_\beta \kappa^{1/2}(\|v\|_2)^{1/2}] \qquad \text{(A.7)}$$

Therefore choosing $\tilde{\delta}$ in such a way that the term of (A.7) is strictly positive we get (A.2). ∎

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REFERENCES

1. N. Alikakos, P. W. Bates, and X. Chen, Convergence of the Cahn–Hilliard equation to the Hele–Shaw model, *Arch. Rat. Mech. Anal.* **128**:165–205 (1994).
2. A. Asselah, Stability of a wave front for a nonlocal conservative evolution, *Proc. Royal Soc. Edinburgh* **128A**:219–234 (1998).
3. K. Kawasaki, Diffusion constants near the critical point for time-dependent Ising models. I. *Phys. Rev.* **145**:224–230 (1966).
4. J. Bricmont, A. Kupiainen, and J. Taskinen, Stability of Cahn–Hilliard fronts, *Comm. Pure & Appl. Math.* **52**:839–871 (1999).
5. E. A. Carlen, M. C. Carvalho, and E. Orlandi, Free energy inequalities and the rate of relaxation to instanton for interfaces profiles in Glauber dynamics, *Nonlinear Differ. Equ. Appl.* **5**:205–218 (1998).
6. E. A. Carlen, M. C. Carvalho, and E. Orlandi, Algebraic rate of decay for the excess free energy and stability of fronts for a non-local phase kinetics equation with a conservation law II, to appear in *Comm. P.D.E.*
7. E. A. Carlen, M. C. Carvalho, and E. Orlandi, Existence, uniqueness and regularity properties for a non-local phase kinetics equation with a conservation law, preprint (1998).
8. X. Chen, Global asymptotic limit of solutions of the Cahn–Hilliard Equation, *J. Diff. Geom.* **44**(2):262–311 (1996).
9. R. Dal Passo and P. de Mottoni, The heat equation with a nonlocal density dependent advection term, preprint (1991).
10. A. De Masi, T. Gobron, and E. Presutti, Travelling fronts in non local evolution equations, *Arch. Rat. Mech. Anal.* **132**:143–205 (1995).

11. A. De Masi, E. Orlandi, E. Presutti, and L. Triolo, Motion by curvature by scaling non local evolution equations, *J. Stat. Phys.* **73**:543–570 (1993).
12. A. De Masi, E. Orlandi, E. Presutti, and L. Triolo, Glauber evolution with Kac potentials I. Mesoscopic and macroscopic limits, interface dynamics, *Nonlinearity* **7**:633–696 (1994).
13. A. De Masi, E. Orlandi, E. Presutti, and L. Triolo, Stability of the interface in a model of phase separation, *Proc. Royal Soc. Edinburgh* **124A**:1013–1022 (1994).
14. A. De Masi, E. Orlandi, E. Presutti, and L. Triolo, Uniqueness of the instanton profile and global stability in non local evolution equations, *Rendiconti di Matematica. Serie VII*, **14** (1994).
15. G. Giacomin and J. Lebowitz, Phase segregation dynamics in particle systems with long range interactions I: Macroscopic limits, *J. Stat. Phys.* **87**:37–61 (1997).
16. G. Giacomin and J. Lebowitz, Phase segregation dynamics in particle systems with long range interactions II: Interface motions, *SIAM J. Appl. Math.* **58**:1707–1729 (1998).
17. R. Glauber, Time-dependent statistics of the Ising model, *J. Math. Phys.* **4**:294–307 (1963).
18. G. Hardy, J. Littlewood, and G. Polya, *Inequalities* (Cambridge University Press, Cambridge, 1932).
19. M. A. Katsoulakis and P. E. Souganidis, Generalized motion by mean curvature as a macroscopic limit of stochastic Ising models with long range interaction and Glauber dynamics, *Comm. Math. Phys.* **169**:61–97 (1995).
20. J. L. Lebowitz, E. Orlandi, and E. Presutti, A particle model for spinodal decomposition, *J. Stat. Phys.* **63**:933–974 (1991).
21. R. L. Pego, Front migration in the nonlinear Cahn–Hilliard equation, *Proc. R. Soc. Lond. A* **422**:261–278 (1989).
22. H. Weyl, Grüppentheorie and Quantenmechanik, Wissenschaftlich Verlag, Leipzig (1926).